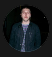
Do Maxwell's equations explain or merely describe electromagnetism?



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Maxwell’s equations individually are just the same ole’ laws that governed Electricity and Magnetism, prior to Maxwell’s modification and realization of the duality of the electromagnetic interaction.

Basically, he took Gauss’s laws for electricity and magnetism, Faraday’s law of induction and Ampere’s law and put them together in a concise and effective manner.

The amazing thing that comes from this, is not the equations as equations, but the consequence of what these equations imply, and the consequence is of course, the coupling of electricity and magnetism in one single fundamental interaction, the Electromagnetic Interaction.

Since, quite recently I revised my rusted electromagnetism, due to a very stupid mistake I made, I would like to elaborate more, mathematically and as an exercise, for the derivation of the Electromagnetic field, from Maxwell’s equations. If you are bored to read this, you can go ahead and skip it, I will just show with maths what I said in the first 3 paragraphs of my answer.

So, basically we have the 4 Maxwell Equations and in the differential form, they are written as:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot B = 0 \tag{2}$$

$$c^2(\nabla \times B) = \frac{J}{\varepsilon_0} + \frac{\partial E}{\partial t} \tag{3}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{4}$$

The 2nd equation, suggests that magnetic charges do not exist i.e magnetic monopoles.

So, here we have our equations. What we want to show now is the relationships in Maxwell’s equations of E and B with respect to potentials and vectors. And we shall begin by solving the 2nd equation, which is the simplest.

The fact that the divergence of B is zero, implies that B is the curl of something (because the divergence of a curl is always zero), so we can rewrite the equation as such:

$$B = \nabla \times A$$

Now, let’s take Faraday’s law:

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

If we express B as derived in equation **(1)**, then we have:

$$\nabla \times E = -\frac{\partial(\nabla \times A)}{\partial t}$$

and since we see that we can differentiate either with respect to space or time first, then we can re-write the equation as:

$$\nabla \times (E + \frac{\partial A}{\partial t}) = 0$$

We now see that the term in the parenthesis, can be represented by a vector $V = E + \frac{\partial A}{\partial t}$ whose curl is equal to 0:

$$\nabla \times V = 0$$

Which means that vector for its curl to be 0, it should be the gradient of something and let’s denote that something as φ.

$$E + \frac{\partial A}{\partial t} = -\nabla\varphi$$

Now, we can apply this in Faraday's law in such a way that when nothing changes with time, the $\frac{\partial A}{\partial t}$ term vanishes and we end up with:

$$E = -\nabla\varphi - \frac{\partial A}{\partial t}$$

Now, we have established that A determines a part of E as well as a part of B.

So, it's a logical question to ask, what would happen if we change A to A', where:

$$A' = A + \nabla\psi$$

what happens to φ, after this is a similar change with respect to ψ:

$$\varphi' = \varphi - \frac{\partial\psi}{\partial t}$$

A consequence of this is that a change in A, or φ does not imply a change in E or B.

So we have two more Maxwell's equations left, and they, as we realize, will give us the relationships between the potentials, the current and the charge density.

So, if we take our second equation (2) and substitute it into Maxwell's first Law, we get:

$$\nabla\cdot(-\nabla\varphi - \frac{\partial A}{\partial t}) = \frac{\rho}{\varepsilon_0}$$

which, after expanding the parenthesis, is equal to:

$$-\nabla^2\varphi - \frac{\partial(\nabla\cdot A)}{\partial t} = \frac{\rho}{\varepsilon_0} \tag{5}$$

and now we have a relationship between φ, A and ρ.

Now, let's move on to try and find the next one, starting from the 4th Maxwell equation.

$$c^2(\nabla\times B) = \frac{J}{\varepsilon_0} + \frac{\partial E}{\partial t},$$

re-arranging yields:

$$c^2(\nabla\times B) - \frac{\partial E}{\partial t} = \frac{J}{\varepsilon_0}$$

and if we substitute for B and E, in terms of what we derived earlier, and after applying the identity that $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$, we end up with:

$$-c^2\nabla^2 A + c^2\nabla(\nabla\cdot A) + \frac{\partial\nabla\varphi}{\partial t} + \frac{\partial^2 A}{\partial t^2} = \frac{J}{\varepsilon_0} \tag{6}$$

…and I really hope that what I intend on writing, is going to actually appear correctly the way I want it to. Although I doubt it and I will probably have to edit the hell out of this post…

Now at this point, it is time we talked about the divergence of A, that we defined earlier. We have the freedom to chose arbitrarily the divergence of it, so we might as well choose it such a way that A and φ are separated but have the same form:

$$\nabla\cdot A = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$$

and if we substitute this, into our third derived equation, the middle terms cancel each other out and after dividing with c^2 we are left with:

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{-J}{\varepsilon_0 c^2} \tag{7}$$

or for φ,

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{8}$$

And now we have expressed Maxwell's Equations in terms of potentials, fields and sources.

In the last two equations, we recognize the Laplacian and if we expand the identity, we have:

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0} \tag{9}$$

And now this equation is so beautiful because it should remind us of how gravitational waves were infered from General Relativity, and also, the solution to this equation looks very similar to the solution of the Schrodinger Equation with the potential U set equal to 0, described by a wave function $\Psi = e^{i(kx-\omega t)}$.

As a side note, quantum mechanics was not yet discovered when Maxwell unified electricity and magnetism and the notion of spin etc was non-existent, so the description of Classical Electrodynamics is now overthrown by Quantum Electrodynamics.

Nevertheless, we can appreciate the magnificence of this set of equations as we can also see the inspiration Maxwell provided to the scientific community!