Solving ODE Symbolically in MATLAB

First Order Equations

We can solve ordinary differential equations symbolically in MATLAB with the built-in M-file *dsolve.*

Example 1. Find a general solution for the first order differential equation

$$y'(x) = xy. \tag{1}$$

We can accomplish this in MATLAB with the following single command, given along with MATLAB's output.

```
>>y = dsolve('Dy = y*x','x')
y = C1*exp(1/2*x^2)
```

Notice in particular that MATLAB uses capital D to indicate the derivative and requires that the entire equation appear in single quotes. MATLAB takes t to be the independent variable by default, so here x must be explicitly specified as the independent variable. Alternatively, if you are going to use the same equation a number of times, you might choose to define it as a variable, say, *eqn1*.

```
>>eqn1 = 'Dy = y*x'
eqn1 =
Dy = y*x
>>y = dsolve(eqn1,'x')
y = C1*exp(1/2*x^2)
```

△

Example 2. Solve the initial value problem

$$y'(x) = xy; \quad y(1) = 1. \tag{2}$$

Again, we use the *dsolve* command.

```
>>y = dsolve(eqn1,'y(1)=1','x')
y =
1/exp(1/2)*exp(1/2*x^2)
```

or

```
>>inits = 'y(1)=1';
>>y = dsolve(eqn1,inits,'x')
y =
1/exp(1/2)*exp(1/2*x^2)
```

△

Now that we've solved the ODE, suppose we want to plot the solution to get a rough idea of its behavior. We run immediately into two minor difficulties: (1) our expression for $y(x)$ isn't suited for array operations (.*, ./, .ˆ), and (2) y, as MATLAB returns it, is actually a symbol (a *symbolic object*). The first of these obstacles is straightforward to fix, using *vectorize()*. For the second, we employ the useful command *eval(),* which evaluates or executes text strings that constitute valid MATLAB commands. Hence, we can use the following MATLAB code to create Figure 1.

```
>>x = linspace(0,1,20);
>>y = eval(vectorize(y));
>>plot(x,y)
```

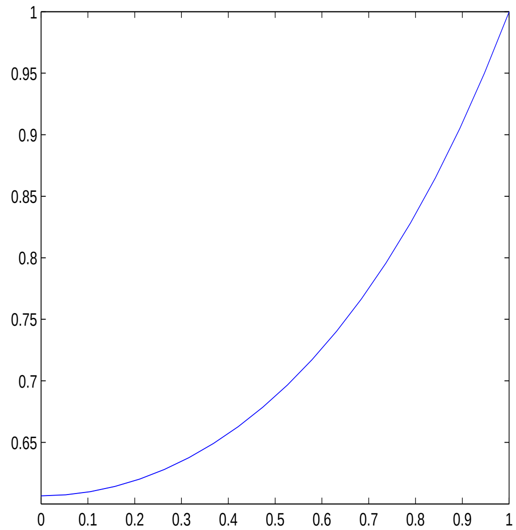

Figure 1: Plot of $y(x) = e^{-1/2}e^{\frac{1}{2}x^2}$ for $x \in [1, 20]$.

Second and Higher Order Equations

Example 3. Solve the second order differential equation

$$y''(x) + 8y'(x) + 2y(x) = \cos(x); \qquad y(0) = 0,\, y'(0) = 1, \tag{3}$$

and plot its solution for $x \in [0, 5]$.

The following MATLAB code suffices to create Figure 2.

```
>>eqn2 = 'D2y + 8*Dy + 2*y = cos(x)';
>>inits2 = 'y(0)=0, Dy(0)=1';
>>y=dsolve(eqn2,inits2,'x')
```

```
y =
exp((-4+14^(1/2))*x)*(53/1820*14^(1/2)-1/130)
+exp(-(4+14^(1/2))*x)*(-53/1820*14^(1/2)-1/130)+1/65*cos(x)+8/65*sin(x)
>>x=linspace(0,5,100);
>>y=eval(vectorize(y));
>>plot(x,y)
```

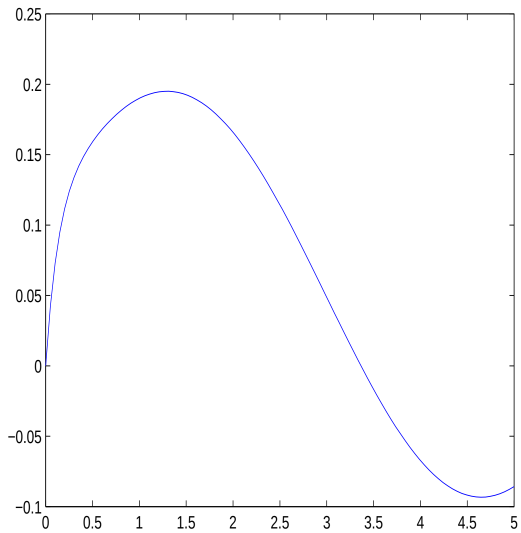

Figure 2: Plot of $y(x)$ for Example 3.

First Order Systems

Suppose we want to solve and plot solutions to the system of three ordinary differential equations

$$\begin{aligned} x'(t) &= x(t) + 2y(t) - z(t) \\ y'(t) &= x(t) + z(t) \\ z'(t) &= 4x(t) - 4y(t) + 5z(t). \end{aligned} \tag{4}$$

First, to find a general solution, we proceed similarly as in the case of single equations, except with each equation now braced in its own pair of (single) quotation marks:

```
>>[x,y,z]=dsolve('Dx=x+2*y-z','Dy=x+z','Dz=4*x-4*y+5*z')
x =
-C1*exp(3*t)-C2*exp(t)-2*C3*exp(2*t)
y =
C1*exp(3*t)+C2*exp(t)+C3*exp(2*t)
z =
4*C1*exp(3*t)+2*C2*exp(t)+4*C3*exp(2*t)
```

Notice that since no independent variable was specified, MATLAB used its default, t. To solve an initial value problem, we simply define a set of initial values and add them at the end of our *dsolve()* command. Suppose we have $x(0) = 1$, $y(0) = 2$, and $z(0) = 3$, and we want to solve the equation for $t \in [0, .5]$. We have, then,

```
inits=’x(0)=1,y(0)=2,z(0)=3’;
[x,y,z]=dsolve(’Dx=x+2*y-z’,’Dy=x+z’,’Dz=4*x-4*y+5*z’,inits)
x =
-5/2*exp(3*t)-5/2*exp(t)+6*exp(2*t)
y =
5/2*exp(3*t)+5/2*exp(t)-3*exp(2*t)
z =
10*exp(3*t)+5*exp(t)-12*exp(2*t)
```

Finally, we can create Figure 3 with the following MATLAB commands.

```
>>t=linspace(0,.5,25);
>>x=eval(vectorize(x));
>>y=eval(vectorize(y));
>>z=eval(vectorize(z));
>>plot(t, x, t, y, ’–’,t, z,’:’)
```

The figure resulting from these commands is included as Figure 3.

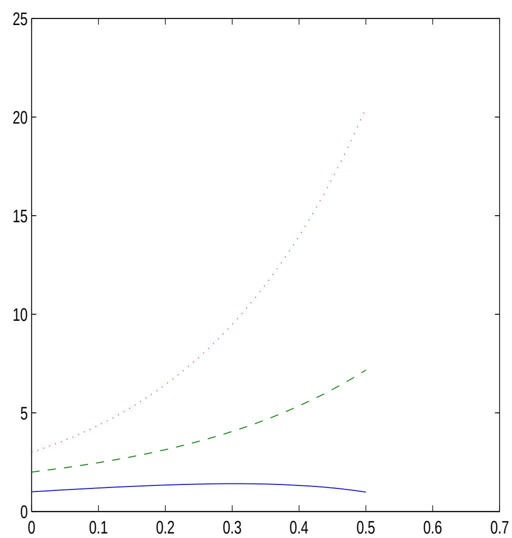

Figure 3: Solutions to equation (4).

Assignments

1. Find a general solution for the differential equation

$$\frac{dy}{dx} = e^y \sin x.$$

2. Solve the initial value problem

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}); \quad p(0) = p_0.$$

3. Solve the initial value problem

$$7y'' + 2y' + y = x; \quad y(0) = 1, y'(0) = 2,$$

and plot your solution for $x \in [0, 1]$.
4. Solve the system of differential equations

$$\begin{aligned} x'(t) &= 2x(t) + y(t) - z(t) \\ y'(t) &= x(t) + 5z(t) \\ z'(t) &= x(t) - y(t) + z(t), \end{aligned}$$

with $x(0) = 1$, $y(0) = 2$, and $z(0) = 3$ and plot your solution for $t \in [0, 1]$.