MATLAB for SMS

Lesson 5: Correlations, Convolutions, and Fourier Transforms

Concepts

The foci of this lesson will be convolutions, Fourier transforms, correlations, and the relationships among them. For the sake of simplicity, we will consider only functions of time, but our logic may be extended to functions of other independent variables as well. The mathematics involved in these extensions are usually trivial, but interpretations of the Fourier 'frequency' become nontrivial in such cases.

Convolution

The *convolution* of two functions $f(t)$ and $g(t)$, where t is defined only on the interval $(0,T)$, is defined as

$$(f * g)(\tau) = \int_0^T f(t)g(t-\tau)dt = \int_0^T g(t)f(t-\tau)dt.$$

Here, the symbol '*' denotes the convolution. Notice that, since t is defined only on $(0,T)$, $f(t)$ and $g(t)$ are also defined only on $(0,T)$. In a form more familiar to SM spectroscopists, the *discrete convolution* is defined by replacing the integration with a summation over a discretized t, with $t_0 = 0$ and $t_N = T$,

$$(f * g)(\tau) = \sum_{i=0}^{N} f(t_i)g(t_i - \tau).$$

Correlation

The *cross-correlation* of two functions is similar to their convolution. The difference arises in the reversal of the time variable. Defining the cross-correlation, $R_{fg}(\tau)$,

$$R_{fg}(\tau) = \int_0^T f(t)g(t+\tau)dt = \int_0^T g(t)f(t+\tau)dt = (f(t) * g(-t))(\tau),$$

we find that it is equivalent to the convolution of the functions $f(t)$ and $g(-t)$. If, however, *either* function $f(t)$ or $g(t)$ is real-valued (and continuous, but we won't be considering continuity much), we find that the cross-correlation and the convolution are equivalent operations under most practical circumstances. e.g.,

$$R_{fg}(\tau) = (f * g)(\tau), \qquad f(t), g(t) \in \mathbb{R}$$

The *autocorrelation* is a special case of the cross-correlation. Specifically, the autocorrelation is the cross-correlation, or convolution, of the function *f(t)* with itself. Under the stipulation that *f(t)* be real-valued, we define the autocorrelation, $R_{ff}(\tau)$.

$$R_{ff}(\tau) = \int_0^T f(t)f(t+\tau)dt = (f * f)(\tau), \qquad f(t) \in \mathbb{R}$$

Fourier Transforms

The Fourier transform decomposes a time-domain signal into its constituent frequencies. The definition provided below illustrates the basic premise that the transformation is the representation of the function *f(t)* by constituent sines and cosines ($e^{i\omega t}$).

$$\hat{f}(\omega) = \int_0^T e^{i\omega t}\, f(t)dt$$

The Fourier transform can be extended to handle discrete input variables, and is known in this form as the discrete Fourier transform. More specifically, when the input variable is discretized time, the transformation is known as the discrete-time Fourier transform. There are several quirks that arise with discrete Fourier transforms, but time will not permit these quirks to be covered in much detail, if at all.

The Convolution Theorem

The convolution theorem relates the convolution of the two functions, f(t) and g(t), to the Fourier representations of each of the functions. Specifically, the convolution theorem states that the Fourier transform of the convolution of *f(t)* and *g(t)*, *(f*g)(τ)*, is equivalent to the pointwise product of the Fourier transforms of *f(t)* and *g(t)*.

$$\hat{f}(\omega)\hat{g}(\omega) = \mathcal{F}[(f * g)(\tau)]$$

In other words, *convolution in the time domain is equivalent to multiplication in the frequency domain.* The inverse operation also holds,

$$(f * g)(\tau) = \mathcal{F}^{-1}\left[\hat{f}(\omega)\hat{g}(\omega)\right],$$

and carries the significance that the convolution (i.e., cross-correlation if our functions are real-valued) of the functions *f(t)* and *g(t)* can be computed with the equation above.

Power Spectra

A power spectrum is often referred to as the spectral density or the power spectral density. It may be interpreted as the "power," as the squared energy per unit frequency, contained within the function *f(t)* at a given frequency ω. The Weiner-Khinchin theorem, a special case of the cross-correlation theorem, relates the power spectrum of the function *f(t)*, $P_{ff}(\omega)$, to the autocorrelation function of *f(t)*, $R_{ff}(\tau)$. Specifically,

$$P_{ff}(\omega) = \mathcal{F}\left[R_{ff}(\tau)\right] = \frac{|\mathcal{F}[f(t)]|^2}{T} = \frac{\left|\hat{f}(\omega)\right|^2}{T}$$

Here, the term in the numerator is the squared amplitude of $\hat{f}(\omega)$, and *T* is the time interval over which *f(t)* is defined. Considering that $\hat{f}(\omega)$ is a complex-valued function of omega, we obtain its amplitude via this relationship:

$$\left|\hat{f}(\omega)\right| = \sqrt{Re\left[\hat{f}(\omega)\right]^2 + Im\left[\hat{f}(\omega)\right]^2}$$

We can therefore directly relate the power spectrum of a function *f(t)* with its autocorrelation function.