MATH 3795
Lecture 14. Polynomial Interpolation.

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Goals

- Learn about Polynomial Interpolation.
- Uniqueness of the Interpolating Polynomial.
- Computation of the Interpolating Polynomials.
- Different Polynomial Basis.

Polynomial Interpolation.

- Given data

x_1	x_2	$\cdots$	x_n
f_1	f_2	$\cdots$	f_n

  (think of $f_i = f(x_i)$) we want to compute a polynomial p_{n-1} of degree at most $n-1$ such that

$$p_{n-1}(x_i) = f_i, \quad i = 1, \ldots, n.$$

- A polynomial that satisfies these conditions is called **interpolating polynomial**. The points x_i are called **interpolation point**s or **interpolation nodes**.

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- A polynomial that satisfies these conditions is called **interpolating polynomial**. The points x_i are called **interpolation points** or **interpolation nodes**.
- We will show that there exists a unique interpolation polynomial. Depending on how we represent the interpolation polynomial it can be computed more or less efficiently.
- Notation: We denote the interpolating polynomial by

$$P(f|x_1, \ldots, x_n)(x)$$

Uniqueness of the Interpolating Polynomial.

Theorem (Fundamental Theorem of Algebra)

Every polynomial of degree n that is not identically zero, has exactly n roots (including multiplicities). These roots may be real of complex.

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Theorem (Uniqueness of the Interpolating Polynomial)

Given n unequal points $x_1, x_2, \ldots, x_n$ and arbitrary values $f_1, f_2, \ldots, f_n$ there is at most one polynomial p of degree less or equal to $n-1$ such that

$$p(x_i) = f_i, \quad i = 1, \ldots, n.$$

Proof.

Suppose there exist two polynomials p_1, p_2 of degree less or equal to $n-1$ with $p_1(x_i) = p_2(x_i) = f_i$ for $i = 1, \ldots, n$. Then the difference polynomial $q = p_1 - p_2$ is a polynomial of degree less or equal to $n-1$ that satisfies $q(x_i) = 0$ for $i = 1, \ldots, n$. Since the number of roots of a nonzero polynomial is equal to its degree, it follows that $q = p_1 - p_2 = 0$. □

Construction of the Interpolating Polynomial.

- Given a basis $p_1, p_2, \ldots, p_n$ of the space of polynomials of degree less or equal to $n-1$, we write

$$p(x) = a_1 p_1(x) + a_2 p_2(x) + \cdots + a_n p_n(x).$$

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- We want to find coefficients $a_1, a_2, \ldots, a_n$ such that

$$\begin{array}{rcccl} p(x_1) & = & a_1 p_1(x_1) + a_2 p_2(x_1) + \cdots + a_n p_n(x_1) & = & f_1 \\ p(x_2) & = & a_1 p_1(x_2) + a_2 p_2(x_2) + \cdots + a_n p_n(x_2) & = & f_2 \\ \vdots & & & & \\ p(x_n) & = & a_1 p_1(x_n) + a_2 p_2(x_n) + \cdots + a_n p_n(x_n) & = & f_n. \end{array}$$

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- This leads to the linear system

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & \dots & p_n(x_1) \\ p_1(x_2) & p_2(x_2) & \dots & p_n(x_2) \\ \vdots & \vdots & & \vdots \\ p_1(x_n) & p_2(x_n) & \dots & p_n(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Construction of the Interpolating Polynomial.

- In the linear system

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & \dots & p_n(x_1) \\ p_1(x_2) & p_2(x_2) & \dots & p_n(x_2) \\ \vdots & \vdots & & \vdots \\ p_1(x_n) & p_2(x_n) & \dots & p_n(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

  if $x_i = x_j$ for $i \neq j$, then the ith and the jth row of the systems matrix above are identical. If $f_i \neq f_j$, there is no solution. If $f_i = f_j$, there are infinitely many solutions.
- We assume that $x_i \neq x_j$ for $i \neq j$.

Construction of the Interpolating Polynomial.

- The choice of the basis polynomials $p_1, \ldots, p_n$ determines how easily

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & \ldots & p_n(x_1) \\ p_1(x_2) & p_2(x_2) & \ldots & p_n(x_2) \\ \vdots & \vdots & & \vdots \\ p_1(x_n) & p_2(x_n) & \ldots & p_n(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

can be solved.

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  can be solved.
- We consider

  Monomial Basis:

$$p_i(x) = M_i(x) = x^{i-1}, \quad i = 1, \dots, n$$

  Lagrange Basis:

$$p_i(x) = L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 1, \dots, n$$

  Newton Basis:

$$p_i(x) = N_i(x) = \prod_{j=1}^{i-1} (x - x_j), \quad i = 1, \dots, n$$

Monomial Basis.

- If we select

$$p_i(x) = M_i(x) = x^{i-1}, \quad i = 1, \ldots, n$$

we can write the interpolating polynomial in the form

$$P(f|x_1, \ldots, x_n)(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \cdots + a_n x^{n-1}$$

- The linear system associated with the polynomial interpolation problem is then given by

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Monomial Basis.

The matrix

$$V_n = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^{n-1} \end{pmatrix}$$

is called the **Vandermonde matrix**.

Monomial Basis.

Example

x_i	0	1	-1	2	-2
f_i	-5	-3	-15	39	-9

For these data the linear system associated with the polynomial interpolation problem is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & -2 & 4 & -8 & 16 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ -15 \\ 39 \\ -9 \end{pmatrix}.$$

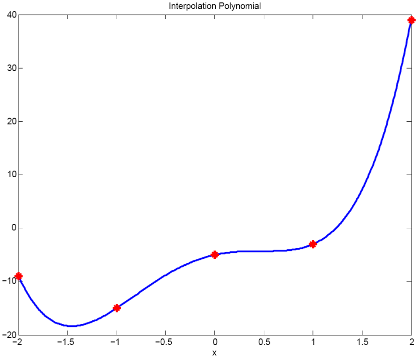
Monomial Basis.

The solution of this system is given by

$$(a_1, a_2, a_3, a_4, a_5) = (-5, 4, -7, 2, 3)$$

which gives the interpolating polynomial

$$P(f|x_1, \dots, x_n)(x)$$
$$= -5 + 4x - 7x^2 + 2x^3 + 3x^4.$$



Horners Scheme.

From

$$p(x) = a_1 + a_2 x + ... + a_n x^{n-1}$$
$$= a_1 + \Big[a_2 + \big[a_3 + [a_4 + \cdots + [a_{n-1} + a_n x] \ldots]x\big]x\Big]x$$

we see that the polynomial represented in the in monomial basis can be evaluated using **Horners Scheme**:

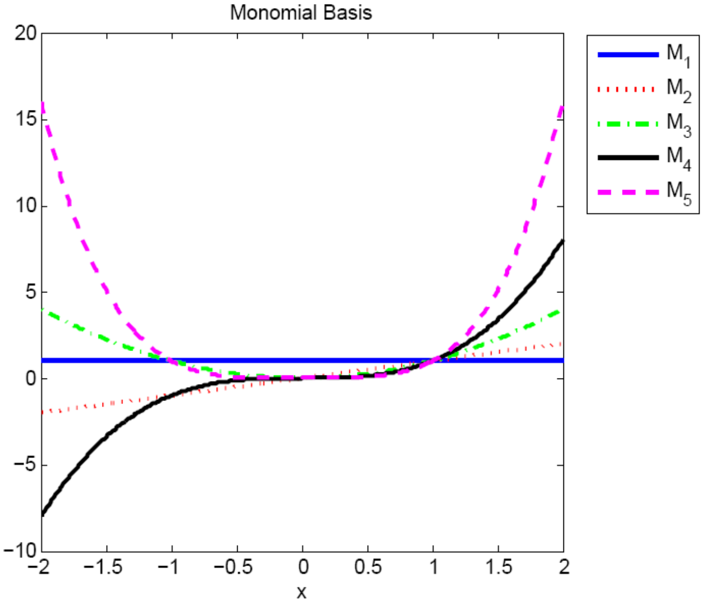
Input: The interpolation points $x_1, \ldots, x_n$.
The coefficients $a_1, \ldots, a_n$ of the polynomial in monomial basis.
The point x at which the polynomial is to be evaluated.
Output: p the value of the polynomial at x.

1. $p = a_n$
2. For $i = n-1, n-2, \ldots, 1$ do
3. $p = p * x + a_i$
4. End

Monomial Basis.

- Computing the interpolation polynomial using the monomial basis, leads to a dense $n \times n$ linear system.
- This linear system has to be solved using the LUdecomposition (or another matrix decomposition), which is rather expensive.
- The system matrix is the Vandermonde matrix, which we have seen in our discussion of the condition number of matrices. The Vandermonde matrix tends to have a large condition number.
- The ill-conditioning of the Vandermonde matrix is also reflected in the plot below, where we observe that the graphs of the monomials $x, x^2, \ldots$ are nearly indistinguishable near $x = 0$.

Monomial Basis.



Lagrange Basis.

- Given unequal points $x_1, \ldots, x_n$, the ith Lagrange polynomial is given by

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

- The Lagrange polynomials L_i are polynomials of degree $n-1$ and satisfy

$$L_i(x_k) = \begin{cases} 1, & \text{if} \quad k = i \\ 0, & \text{if} \quad k \neq i \end{cases}$$

Lagrange Interpolating Polynomial.

- With the basis functions $p_i(x) = L_i(x)$, the linear system associated with the polynomial interpolation problem is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

- The interpolating polynomial is given by

$$P(f|x_1, \ldots, x_n)(x) = \sum_{i=1}^{n} f_i L_i(x)$$

Lagrange Interpolating Polynomial.

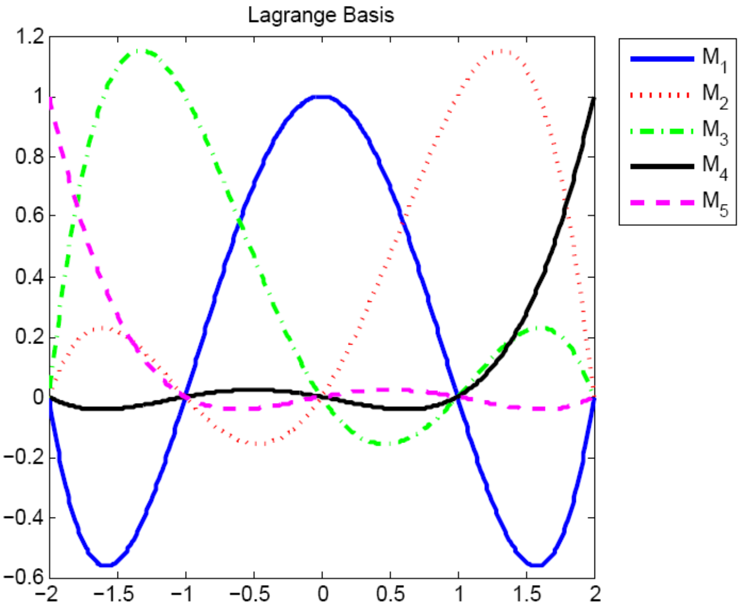
Example

x_i	0	1	-1	2	-2
f_i	-5	-3	-15	39	-9

Interpolation polynomial

$$
\begin{aligned}
&P(f|x_1,\dots,x_5)(x) \\
&= -5 + 4x - 7x^2 + 2x^3 + 3x^4 \quad \text{Monomial basis} \\
&= -5\frac{(x-1)(x+1)(x-2)(x+2)}{4} \\
&- 3\frac{x(x+1)(x-2)(x+2)}{-6} \\
&- 15\frac{x(x-1)(x-2)(x+2)}{-6} \\
&+ 39\frac{x(x-1)(x+1)(x+2)}{24} \\
&- 9\frac{x(x-1)(x+1)(x-2)}{24} \quad \text{Lagrange basis.}
\end{aligned}
$$

Lagrange Basis.



Newton Basis.

- The Newton polynomials are given by

$$N_1(x) = 1, \quad N_2(x) = x - x_1,$$

$$N_3(x) = (x - x_1)(x - x_2), \ldots, N_n(x) = \prod_{j=1}^{n-1}(x - x_j).$$

- N_i is a polynomial of degree $i - 1$. They satisfy $N_i(x_j) = 0$ for all $j < i$.
- With the basis functions $p_i(x) = N_i(x)$, the corresponding matrix associated with the polynomial interpolation problem is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & x_2 - x_1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & x_{n-1} - x_1 & \ldots & \prod_{j=1}^{n-2}(x_{n-1} - x_j) & 0 \\ 1 & x_n - x_1 & \ldots & \prod_{j=1}^{n-2}(x_n - x_j) & \prod_{j=1}^{n-1}(x_n - x_j) \end{pmatrix}$$

Newton Basis.

The system matrix is lower triangular. If all interpolation nodes $x_1, \dots, x_n$ are unequal, then the diagonal entries of the system matrix in are nonzero and we can compute the coefficients by forward substitution,

$$
\begin{aligned}
a_1 &= f_1 \\
a_2 &= \frac{f_2 - a_1}{x_2 - x_1} \\
&\vdots \\
a_n &= \frac{f_n - \sum_{i=1}^{n-1} a_i \prod_{j=1}^{i-1}(x_n - x_j)}{\prod_{j=1}^{n-1}(x_n - x_j)}
\end{aligned}
$$

Newtone Interpolating Polynomial.

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Interpolation polynomial

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\begin{aligned}
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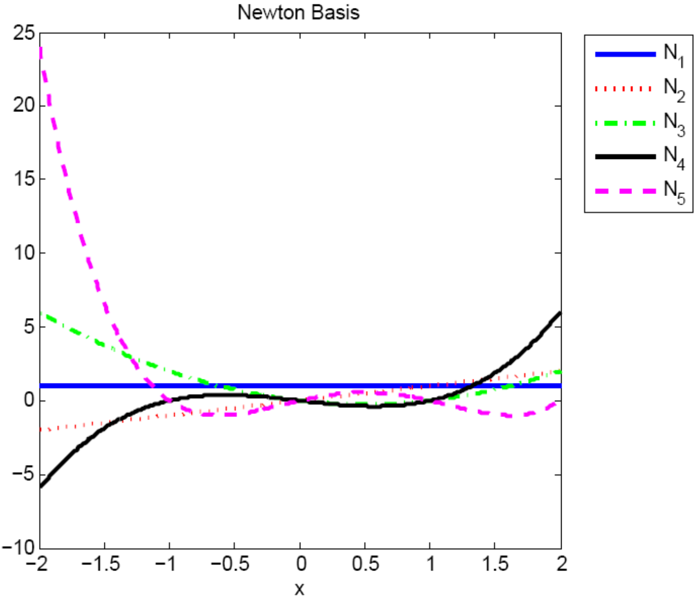
&= -5 + 4x - 7x^2 + 2x^3 + 3x^4 \quad \text{Monomial basis} \\
&= -5\frac{(x-1)(x+1)(x-2)(x+2)}{4} - 3\frac{x(x+1)(x-2)(x+2)}{-6} \\
&- 15\frac{x(x-1)(x-2)(x+2)}{-6} + 39\frac{x(x-1)(x+1)(x+2)}{24} \\
&- 9\frac{x(x-1)(x+1)(x-2)}{24} \quad \text{Lagrange basis} \\
&= -5 + 2x - 4x(x-1) + 8x(x-1)(x+1) + 3x(x-1)(x+1)(x-2)
\end{aligned}
$$

Newton basis.

Newton Basis.



Construction of the Interpolating Polynomial. Summary.

- If $x_i \neq x_j$ for $i \neq j$, there exists a unique polynomial of degree $n-1$, denoted by $P(f|x_1, \ldots, x_n)(x)$ such that

$$P(f|x_1, \ldots, x_n)(x_i) = f_i, \quad i = 1, \ldots, n.$$

- The interpolating polynomial can be written in different bases:

$$\begin{aligned}
&P(f|x_1, \ldots, x_n)(x) \\
&= a_1^M + a_2^M x + \cdots + a_n^M x^{n-1} \\
&= f_1 \prod_{\substack{j=1 \\ j\neq 1}}^{n} \frac{x - x_j}{x_1 - x_j} + f_2 \prod_{\substack{j=1 \\ j\neq 2}}^{n} \frac{x - x_j}{x_2 - x_j} + \cdots + f_n \prod_{\substack{j=1 \\ j\neq n}}^{n} \frac{x - x_j}{x_n - x_j} \\
&= a_1^N + a_2^N (x - x_1) + \cdots + a_n^N (x - x_1) \ldots (x - x_{n-1}).
\end{aligned}$$

- The representation of the interpolating polynomial depends on the chosen basis.