Maximum inductance for given wire length

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The question as given in the *Morseman* column, one which was recently set in a first-year University Engineering test, was

"You are given a fixed length of wire. You have to wind it into a coil having the *largest possible* inductance. Should you wind a short, fat coil, or a long, thin coil?"

The answer was meant to be reasoned from the definition of inductance, which is proportional to the total linked flux within a coil. It goes like this:

> "The total linked flux in a coil is proportional to the product of the magnetic field in each turn, B, the area of one turn, A, and the number of turns, n, that is, proportional to BAn.
>
> "If the radius is doubled, the area A increases by a factor of 4, and the number of turns, n, decreases by a factor of 2, because our wire length is constant. Therefore the linked flux increases by a factor of $4/2 = 2$.
>
> "Therefore we will get maximum inductance of we wind a short, fat, coil, where the loop area is as large as possible".

The fishhook in this question is that you are then tempted to reason like this:

> "Increasing the loop area has a greater effect on inductance than increasing the number of turns. Therefore, maximum inductance will be obtained for a *single turn* coil, which has the maximum loop area for a given length of wire."

We can demonstrate this reasoning analytically by applying it to the expression for the inductance of a "long solenoid", which the students interrogated by the question have just met in lectures. (This is always the first formula derived in any treatment of inductance).

The inductance of a long solenoid is readily shown by applying Faraday's and Ampere's Laws to be

$$L = \mu_0 N^2 A b \tag{1}$$

$$\text{where } \mu_0 = 4\pi \times 10^{-7} \text{ Henry/metre} = \text{ permeability of free space} \tag{2}$$

$$N = \text{ number of turns per metre (not total turns!)} \tag{3}$$

$$A = \text{ cross-sectional area of coil, metres} \tag{4}$$

$$b = \text{ length of coil, metres} \tag{5}$$

The number of turns and the coil length are constrained by the fixed length of wire that we have.

$$\text{Let } s = \text{ total length of wire} \tag{6}$$

$$\text{then } n = \frac{s}{2\pi a} = \text{ total turns in coil} \tag{7}$$

$$\text{but also } n = Nb \tag{8}$$

$$\text{so } N = \frac{n}{b} = \left(\frac{s}{2\pi a}\right)\frac{1}{b} = \frac{s}{2\pi ab} \tag{9}$$

$$\text{and } A = \pi a^2 \tag{10}$$

$$\text{substituting into equation (1)} \quad L = \mu_0 \left(\frac{s^2}{4\pi^2 a^2 b^2}\right)\left(\pi a^2\right) b \tag{11}$$

$$L = \frac{\mu_0 s^2}{4\pi b} \tag{12}$$

Equation 12 indicates that *maximum* inductance will be obtained when b is as *small as possible.* In the limit, this will occur when the loop has a single turn, as reasoned above.

But from physical considerations, this argument is flawed. Equation 1 no longer holds. It is derived assuming that the magnetic field, B, is *constant* over the surface area of the coil, and inside it, and is zero everywhere outside. These assumptions are *approximately* true for a *long* solenoid, but are incorrect for a single turn loop.

The field around a single loop (draw some field lines to see this) is not negligible. Furthermore, B varies across the loop. The axial magnetic field at the centre of a single circular loop of radius a carrying current I, due to all sections of the wire, is

$$B = \frac{\mu_0 I}{2R} \tag{13}$$

Off axis, as we approach very close to the wire, the field is influenced *only* by the *nearest* portion of the wire. At a distance r from the wire, it becomes[1]

$$B = \frac{\mu_0 I}{2\pi r} \tag{14}$$

$$\text{where } r = \text{the radial distance from centre of the wire} \tag{15}$$

These expressions are different. Furthermore, at the wire surface, r becomes the wire radius, which is very much smaller than R, so the field here is much higher than that in the centre. Thus the "uniform" assumption made in the "long solenoid" approximation is also invalid.

The "short and fat is better" claim will still be true when the coil is "adequately long". But since the equation on which it's based breaks down in the limit of a single-turn coil, we can't base an absolute argument on it.

Is there a point at which "shortness and fatness" ceases to be a bonus and becomes a liability? This is certainly true for humans, and we will see that it's also true for coils.

Thus, a better question to ask would be, "given a constant length of wire, is there a *particular ratio* of length to radius which gives maximum inductance?'

A More Accurate Equation

Equation 1 is never used to describe the behaviour of real coils, it's too simpleminded. The standard one conventionally used by Hams and engineers is *Wheeler's 1928 formula*, given below. We will see that this does *not* predict the answer reasoned above.

Wheeler's 1928 formula was claimed by him to be "a close approximation for single-layer coils having length greater than 0.4 times the radius." It is

$$L = \frac{a^2 n^2}{9a + 10b} \tag{16}$$

$$\text{where } L = \text{inductance in microHenry} \tag{17}$$

$$a = \text{coil radius, inches,} \tag{18}$$

$$b = \text{coil length, inches,} \tag{19}$$

$$n = \text{total number of turns} \tag{20}$$

[1]Both of these expressions are proved in standard texts, including mine, *Electromagnetism*, Published by the University of Auckland Physics Department, \$15, chapter 6.

This formula has also been converted to SI units (see below). However, we will use this original form - the units don't matter - and insert similar substitutions to see what it predicts.

$$\text{Let } s = \text{given length of wire,} \tag{21}$$
$$d = \text{centre-to-centre spacing of windings.} \tag{22}$$
$$\text{then } 2\pi a = \text{length in one turn of the coil.} \tag{23}$$
$$n = \frac{s}{2\pi a} \tag{24}$$
$$b = \frac{sd}{2\pi a} \tag{25}$$

Consider the numerator of equation 16:

$$a^2 n^2 = \left(a^2\right)\left(\frac{s^2}{4\pi^2 a^2}\right) \tag{26}$$
$$= \frac{s^2}{4\pi^2} = \text{constant.} \tag{27}$$

Thus the numerator of equation 16 is invariant for a constant wire length, so we need only consider variations in the denominator, D.

$$D = 9a + 10b = 9a + 10\left(\frac{sd}{2\pi a}\right) \tag{28}$$
$$= 9a + 10\left(\frac{sd}{2\pi}\right)\frac{1}{a} \tag{29}$$
$$D = 9a + \frac{10k}{a} \qquad \text{where } k = \frac{sd}{2\pi} \tag{30}$$

The standard technique for finding the maximum or minimum of a function is to use calculus. Differentiate equation 30 with respect to a. This gives us the *gradient* of the function. Equate this to zero and solve for a. This gives the value of a where the gradient is zero, which must be either a maximum or a minimum.[2] Divide the corresponding value of b by this value of a. This gives the ratio of length to diameter for maximum inductance.

$$\text{differentiating, } \frac{dD}{da} = 9 - \frac{10k}{a^2} = 0 \tag{31}$$
$$\text{solving, } a^2_{max} = \frac{10k}{9} \tag{32}$$
$$\text{but } b = \frac{sd}{2\pi a} = \frac{k}{a} \tag{33}$$
$$\text{so } k = ab \tag{34}$$
$$\text{from equation (32), } a^2_{max} = \frac{10\, a_{max}\, b_{max}}{9} \tag{35}$$
$$\text{whence } \left(\frac{b}{a}\right)_{max} = 0.9 \tag{36}$$

Thus, Wheeler's simple formula predicts that maximum inductance results when the length of the coil is 0.9 times the radius. This is within Wheeler's region of "close approximation". And it directly contradicts the limiting argument of the first section.

I know of no other derivation of this, but Alan Sharp's webpage states that it's "possible to show it" without giving a source, so it's been done somewhere. Alan then shows a plot which is similar to the first one of mine below. See the link at

[2] It can be shown to be a minimum, which we want, by differentiating again and observing that $\partial D^2/\partial^2 a$ is negative.

htpp://members.aol.com/alansharp03/homepage.htm

Several versions of Wheeler's simple formula have also been given in SI units. One published form is

$$L = \frac{d^2 n^2}{l + 0.45d} \quad (37)$$

$$\text{where now } d = \text{coil diameter in metres} \quad (38)$$

$$\ell = \text{coil length in metres.} \quad (39)$$

$$n = \text{number of turns} \quad (40)$$

This expression gives a result which is 1.6% lower than the original version given in inches, but in practice they're essentially the same. One reference to this SI version states that "this formula is accurate within 1 % for $\ell > 0.4d$, but I dispute this, as, by inference, does Wheeler. See the plots under "Numerical comparisons" below.

Verification from Wheeler's Simple Formula

Figure 1, top, shows inductance in μH, calculated with Wheeler's 1928 formula, against coil radius in inches for a 200 inch length of wire 1/16 inch in diameter. Close inspection shows that the maximum inductance is 120 μH, occurring at a coil radius of 0.47 inches.

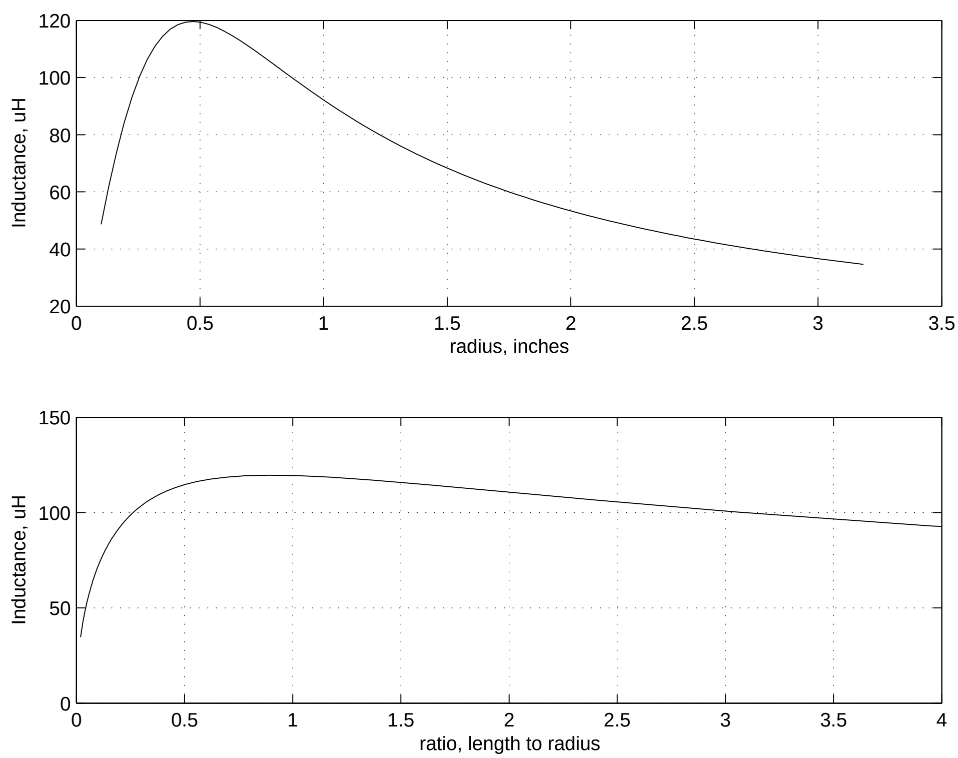

Figure 1: Top: Inductance as a function of coil radius. Bottom: Inductance as a function of length/diameter.

The bottom plot shows the inductance plotted against the ratio of length/diameter, b/a. Close inspection shows that there is indeed be a broad maximum of inductance centred on $b/a = 0.9$.

An Even More Accurate Equation

In 1982 Wheeler published another formula, which he maintained had "a relative error of less than 0.001". This was[3]

$$L = \mu_0 n^2 a \left[\log_e \left(1 + \frac{\pi a}{b}\right) + \frac{1}{2.3 + 1.6\left(\frac{b}{a}\right) + 0.44\left(\frac{b}{a}\right)^2} \right] \tag{41}$$

This reduces to equation (1) as $b \to \infty$. To see this, observe that the second term in the square brackets tends to zero as b increases, and the first term involving the logarithm simplifies:

$$\text{since } \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \tag{42}$$

$$\approx x \qquad \text{when } x \text{ is small} \tag{43}$$

$$\text{so } \log_e\left(1 + \frac{\pi a}{b}\right) \approx \frac{\pi a}{b} \tag{44}$$

$$\text{and } L \approx \mu_0 n^2 a \left(\frac{\pi a}{b}\right) = \mu_0 \left(\frac{n^2}{b^2}\right) \pi a^2 b \tag{45}$$

$$L \approx \mu_0 N^2 A b \tag{46}$$

Equation 46 is seen to be the inductance of a long solenoid. The expansion of the logarithm and the addition of the second term give corrections to compensate for changing coil geometry. Wheeler's 1928 equation is just a simplified form of this which holds over a smaller region of geometries.

At the other asymptotic extreme, we can check L using this accurate formula for a single turn coil. After some algebra, the equation reduces to

$$L = \mu_0 a \left[\log_e \left(\frac{8a}{b}\right) - 0.5 \right] \tag{47}$$

which Grover[4] gives as the correct expression for a single turn of very thin metal tape. He gives the correct expression for a circular *wire*, arrived at by more fundamental methods, as

$$L = \mu_0 a \left[\log_e \left(\frac{8a}{b}\right) - 1.75 \right] \tag{48}$$

Since this is correct in these two extreme cases (or nearly so in the second), Wheeler's accurate formula seems justifiable as a very good approximation for *all* intermediate geometries, as he claimed.

But equation 48 is interesting, because the coil length, b, is now just the diameter of the wire. As the wire diameter tends to zero (finer wire), the inductance tends to infinity! This can be shown to follow from physical considerations. As we have seen above, the B field at the wire surface is inversely proportional to the wire radius, and thus *also* tends to infinity as radius decreases. And it turns out that this causes a steadily increasing contribution to the magnetic flux near the surface of the wire, leading to an increase in inductance.

[3]Harold A. Wheeler, *Inductance formulas for circular and square coils*, Proc IEEE, (**70**), **number 12, December 1982, pp 1449 - 1450.**

[4]Frederick C. Grover *Inductance calculations, formulae and tables*, Dover, 1946, p. 143

There is a transcript of an interesting interview with Wheeler by Ronald R. Kline, of the IEEE History Center, on August 28, 1985 at

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http://www.ieee.org/portal/cms_docs_iportals/iportals/
aboutus/history_center/oral_history/pdfs/Wheeler048.pdf
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That URL should all be on one line, but I had to break it to fit the page. Alternatively, you can find it by Googling on "Wheeler interview inductance".

In this interview, Wheeler also discusses a whole variety of other problems that he attacked, including the Neotrodyene, and AVC systems. When he talks about his many "inductance" publications, he makes the point that he continually attempted to present *simple* expressions which were useful for *slide-rule computation*. The terms in them didn't relate directly to physical observables, but were the result of empirical best-fits to the tables and equations which did.

The 1982 equation is of this type. He infers that he started from expressions that gave correct results in the limit of very long and very short coils, then juggled other parameters to give good fits for intermediate geometries.

Differentiation of Wheeler's Second formula

In principle, we can differentiate equation 41 as before, find the value of a that makes it zero, find the corresponding value of b, and hence the value of b/a for maximum inductance. First, we eliminate n and b (which are both functions of a) and absorb the constant μ_0 by defining

$$k_1 = \frac{\mu_0 s^2}{4\pi^2 a} \tag{49}$$

$$k_2 = \frac{sd}{2\pi}. \tag{50}$$

Now differentiating [5] equation 41 and equating the result to zero we find that

$$\frac{dL}{da} = \frac{k_1}{a}\left[\frac{2\pi a}{\left(\frac{\pi a^2}{k_2}+1\right)k_2} + \frac{\frac{1.76k_2^2}{a^5}+\frac{3.2k_2}{a^3}}{\left(\frac{0.44k_2^2}{a^4}+\frac{1.6k_2}{a^2}+2.3\right)^2}\right] \tag{51}$$

$$-\frac{k_1}{a^2}\left[\frac{1}{\frac{0.44k_2^2}{a^4}+\frac{1.6k_2}{a^2}+2.3} + \log_e\left(1+\frac{\pi a^2}{k_2}\right)\right] = 0 \tag{52}$$

This is far too frightening to solve analytically. I won't even attempt it. Instead, we will look at the results numerically.

Numerical Comparison of the Predictions of Wheeler's Equations

For a practical comparison, I again assumed 200 inches of closewound wire having a diameter of 1/16 inch. Figure 2 shows the inductance of the resulting coil calculated with both formulae as a function of b/a.

[5] This equation is error-prone to differentiate analytically, so I used a freeware algebra program, *wxMaxima*, that you can download from the web. It can also integrate, find power series etc.

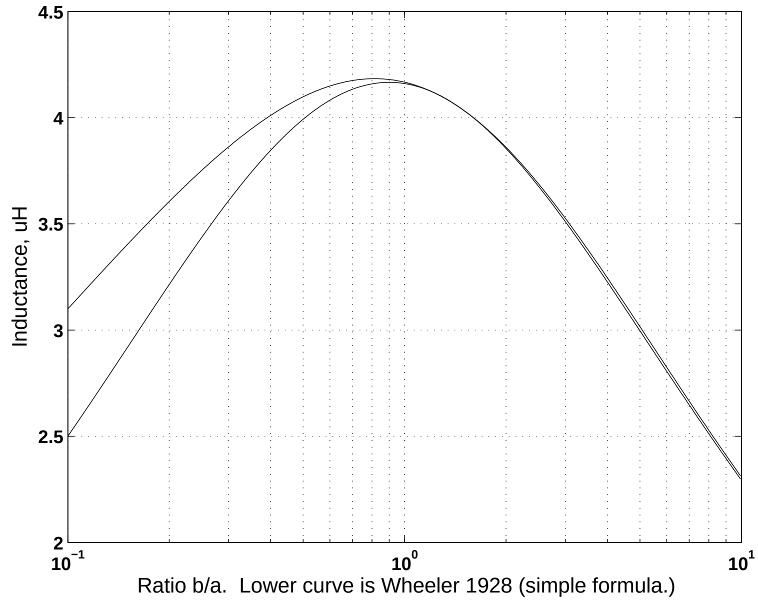

Figure 2: Comparison of maximum inductance predictions from Wheeler's two equations: Top: Accurate, 1982 formula. Bottom: Simple, 1928 formula.

The right-hand, downward-trending portions of both plots are almost identical, showing that as the coil becomes "longer and thinner", the simple 1928 formula is as good as the later, 1982 one. But the plots diverge markedly in the left-hand region, where the 1982 formula is expected to be more accurate. Wheeler's accuracy criterion for the 1928 formula specified that b/a should be greater than 0.4, but we can see that at this value the two plots are perceptibly different, with the "accurate" prediction about 4% greater than the "simple" prediction. Curve shapes for a wide variety of geometries, not shown here, show the same features.

It is seen that maximum inductance occurs at a slightly different value of b/a for each curve. The "accurate" formula predicts, by inspection, a value of $b/a \approx 0.814$[6] over a wide range of wire lengths and turn spacings, as opposed to 0.9 predicted by the simple formula. But both the maxima are very broad, so it doesn't matter much.

Conclusions

There is indeed a "magic value" of coil length to coil radius, which, for a given length of wire, results in maximum inductance. The resulting value of maximum inductance depends upon the wire length, but the ratio is independent of it. The ratio determined from Wheeler's simple formula is exactly 0.9, that from his more complicated formula approximately 0.814. The maxima is broad.

Historically, inductance was the last of the three fundamental electromagnetic circuit properties to be understood. It was not even *called* "inductance" until Oliver Heaviside coined the term in 1886, long after "resistance" and "capacitance" had been defined. It is also, for the reasons given above, much more difficult to calculate than the other two, and much more difficult to measure accurately[7], since magnetic field lines extend outside the coil, and hence interact with the environment.

[6] the ratio is closer to 0.815 for short wire lengths (fewer turns) , closer to 0.813 for long lengths (more turns),

[7] although in calculating the inductance of a long solenoid we pretend that they don't.

References

Several on-line inductance calculators are available on the web. Some quote the formulae used, others don't. I give URLs below. To compare their predictions, I used each to compute the inductance of a coil of 50 turns, having radius and length 5 cm. The answers I got are underneath each.

For this coil, Wheeler's simple formula gives 259.01 μH.
Wheeler's accurate formula gives 259.42 μH.

`http://hamwaves.com/antennas/inductance.html`

This is a very sophisticated calculator, and also gives Q, if you specify some other variables. It gives $L = 256.978\,\mu$H (with 0.5 mm diameter copper-plated wire assumed).

I suspect that this is the most accurate calculator, since it's accompanied by extensive commentary and technical references, and includes corrections for a variety of parameters. This page contains much useful information, including advice on which published calculators to avoid because they are flawed.

`http://www.crystalradio.net/cal/indcal2.shtml`
$L = 259.013\,\mu$H

`http://my.athenet.net/~multiplx/cgi-bin/airind.cgi`
$L = 259.2243\,\mu$H

`http://www.qsl.net/wa2whv/radiocalcs.shtml`
$L = 258.2760\,\mu$H

`http://www.mogami.com/e/cad/coil-01.html`
$L = 259.3288\,\mu$H

References to various online programs and downloadable codes are at

`http://www.circuitsage.com/inductor.html`

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