Design of Planar Rectangular Microelectronic Inductors

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Abstract—Negative mutual inductance results from coupling between two conductors having current vectors in opposite directions. As a quantity in electronic circuits, negative mutual inductance is usually so much smaller in magnitude than overall inductance that it can be neglected with little effect. In the microelectronic world, however, its neglect can result in inductance values as much as 30 percent too high. This paper derives inductance equations for planar thin- or thick-film coils, comparing equations that include negative mutual inductance with those that do not. It describes a computer program developed for calculating inductances for both square and rectangular geometries, the variables considered being track width, space between tracks, and number of turns. Graphic results are presented for up to 16 turns over an inductance range of 3 nanohenries to 10 microhenries. Although details of fabrication are not included, the effects of film thickness and frequency on the mutual-inductance parameter are discussed.

INTRODUCTION

Technological progress in the areas of hybrid microelectronics and microwave integrated circuits during the past decade has seen thin-film microelectronic inductors used to an ever-increasing extent. Inductor design throughout this period, reflected in the technical literature [1]-[3] has been based largely on older theories and derivations, some dating back 100 years [4]. Now, as these inductors become smaller, the assumptions that have governed their design in the past become less valid. Nevertheless, inductor design, artwork preparation, photoreduction, and fabrication are time-consuming processes, and redesign and reprocessing must be kept to a minimum. Graphic representations of computer-made complex inductance calculations are an invaluable means toward this end.

BASIC MATHEMATICAL CONCEPTS

Self-Inductance Calculations for Straight Conductors

All theoretical equations for calculations involving planar rectangular inductors having one or more turns employ in their derivation the self-inductance of a straight conductor. The exact self-inductance for a straight conductor is [1]

$$L = 0.002\ell[\ln(2\ell/\mathrm{GMD}) - 1.25 + \mathrm{AMD}/\ell + (\mu/4)T] \quad (1)$$

where L is the inductance in microhenries, ℓ is the conductor length in centimeters, GMD and AMD represent the geometric and arithmetic mean distances, respectively, of the conductor cross section, μ is the conductor permeability, and T is a frequency-correction parameter.

The geometric mean distance (GMD) between two conductors is the distance between two infinitely thin imaginary filaments whose mutual inductance is equal to the mutual inductance between the two original conductors. The GMD of a conductor cross section is the distance between two imaginary filaments normal to the cross section, whose mutual inductance is equal to the self-inductance of the conductor.[2] By definition, the self-inductance of a conductor is the sum of the mutual inductances of all the pairs of filaments of which it is composed. The GMD is equal to 0.7788 times the radius in the case of a circular cross section, 0.44705 times a side in the case of a square cross section, and 0.22313 times the length in the case of a straight-line cross section. Computation of the GMD for a rectangular cross section is lengthy, but its value—which is a function of the ratio between sides a and b—is easily seen to lie within a narrow range: for the limiting case where $b \to 0$, the value is $0.22313\,(a+b)$; for the limiting case where $a=b$, the value is $0.22352(a+b)$.

The arithmetic mean distance is the average of all the distances between the points of one conductor and the points of another. For a single conductor, the arithmetic mean distance is the average of all possible distances within the cross section. In the case of a circular cross section, the AMD equals the radius; in the case of a straight-line cross section, the AMD equals one-third the length. Thin-film conductors approach the straight-line condition: as the film thickness approaches 0, the AMD of a thin-film track approaches one-third the width.

If GMD and AMD values for a circular cross section are substituted into (1), we obtain

$$\begin{aligned} L &= 0.002\ell[\ln(2\ell/0.7788r) - 1.25 + r/\ell + (\mu/4)T] \\ &= 0.002\ell[\ln(2\ell/r) - \ln 0.7788 - 1.25 + r/\ell + (\mu/4)T] \\ &= 0.002\ell[\ln(2\ell/r) - 1 + r/\ell + (\mu/4)T] \end{aligned} \quad (2)$$

which is the exact equation for a circular cross section, r being the radius. For the near-direct-current condition, T equals 1 and the equation becomes

$$L = 0.002\ell[\ln(2\ell/r) - 1 + r/\ell + \mu/4]. \quad (3)$$

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[1]Though not directly stated in the literature, this equation is easily derived by combining equations (6), (8), and (211) of Grover (see reference 5). The value of T, which varies from 1 at direct current to 0 at infinite frequency, can be found for a conductor of circular cross section from Table 52, page 266 of Grover.

[2]The concept of cross-section geometric mean distance goes back to Maxwell's examples in article 692, Volume II of reference 4.

If the conductor has a magnetic permeability of 1, (3) reduces to

$$L = 0.002\ell[\ln(2\ell/r) - 0.75 + r/\ell] \quad (4)$$

and if the length is many orders of magnitude greater than the radius, it becomes

$$L = 0.002\ell[\ln(2\ell/r) - 0.75]. \quad (5)$$

Equations (3), (4), and (5) are supported by most authoritative sources [5]-[7].

For thin-film inductors with rectangular cross sections, (1) takes the form

$$\begin{aligned} L &= 0.002\ell\left\{\ln[2\ell/0.2232(a+b)] - 1.25 + [(a+b)/3\ell] + (\mu/4)T\right\} \\ &= 0.002\ell\left\{\ln[2\ell/(a+b)] - \ln 0.2232 - 1.25 + [(a+b)/3\ell] + (\mu/4)T\right\} \\ &= 0.002\ell\left\{\ln[2\ell/(a+b)] + 0.25049 + [(a+b)/3\ell] + (\mu/4)T\right\} \end{aligned} \quad (6)$$

where a and b are the rectangular dimensions of the cross section. For the near-direct-current case in which magnetic permeability is 1, (6) reduces to[3]

$$L = 0.002\ell\left\{\ln[2\ell/(a+b)] + 0.50049 + [(a+b)/3\ell]\right\}. \quad (7)$$

As Table I indicates, the skin-depth phenomenon has little effect on thin films, and T in (6) should be considered to have a value of 1 for microwave frequencies. For thicker films and lower frequencies, corrections may be required and must be considered [5], [7], [9].

TABLE I
Variations in Frequency-Correction Parameter T for Thin Films and Microwave Frequencies

Value of T	Film Thickness	Frequency
0.9974	10,000 angstroms	10 gigahertz
0.9986	0.0025 millimeter (0.1 mil)	1 gigahertz
0.9095	0.0075 millimeter (0.3 mil)	1 gigahertz

Mutual-Inductance Calculations for Planar Coils

In the case of an L-shaped thin-film inductor, total inductance is equal to the sum of the self-inductances of the two straight segments and is less than the inductance of a single straight track of equal total length. In the case of a rectangular or square planar coil, straight conductor segments parallel other straight conductor segments and the mutual inductance between these parallel tracks contributes to the total inductance of the coil.

Fig. 1 illustrates the mutual inductance $M_{1,2}$ that results from a singularly generated current i_1. Here,

$$M_{1,2} = d\phi_{1,2}/di_1$$

[3]Equation (7) is in substantial agreement with equations derived by others (cf. [5] through [9]), the difference being that they have assumed a square cross section whereas we have assumed a rectangular cross section in which one side is many times greater than the other.

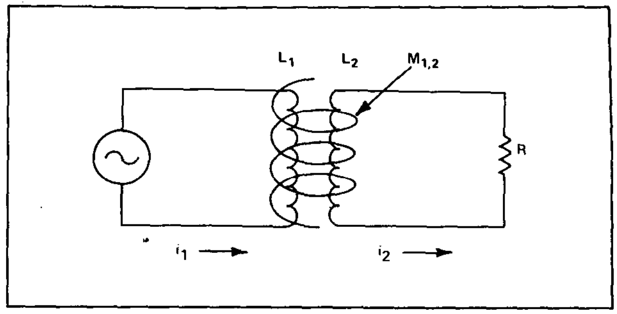

Fig. 1. Mutual inductance resulting from singularly generated current.

where $\phi_{1,2}$ is the flux common to self-inductances L_1 and L_2 that is caused by the generated current, i_2 being the induced current. Fig. 2 illustrates the mutual inductance that results from two generated currents, i_1 and i_2. In this case,

$$M_{1,2} = d\phi_{1,2}/di_1$$

and

$$M_{2,1} = d\phi_{2,1}/di_2$$

where $\phi_{1,2}$ is the flux common to self-inductances L_1 and L_2 that is caused by current i_1, and where $\phi_{2,1}$ is the flux common to self-inductances L_1 and L_2 that is caused by current i_2. When the frequencies of the two current generators are the same, the total mutual inductance M_T is equal to the vector sum of $M_{1,2}$ and $M_{2,1}$; when these frequencies differ, the instantaneous sum must be used.

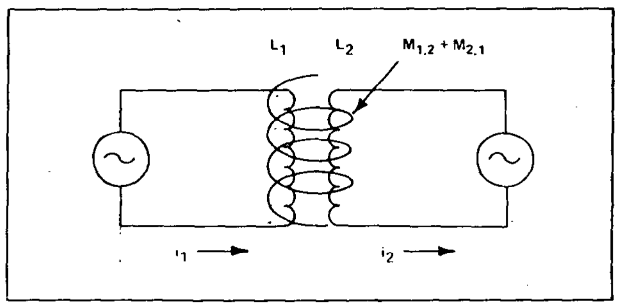

Fig. 2. Mutual inductance resulting from two generated currents.

Consider the case of the two-turn planar rectangular coil represented schematically in Fig. 3. The total inductance of this coil is equal to the sum of the self-inductances of each of the straight segments ($L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8$) plus all the mutual inductances between the segments. The mutual inductance between segments 1 and 5 has a component $M_{1,5}$ cuased by the current flowing in segment 1, and a component $M_{5,1}$ caused by the current flowing in segment 5. Since the frequency and phase in both segments are identical, the total mutual inductance linking them equals $M_{1,5}+M_{5,1}$. An analogous relationship exists between segment pairs 2-6, 3-7, and 4-8; in each of these pairs, current flow is in the same direction in both segments and all mutual inductances are positive. The mutual inductance between segments 1 and 7, on the other hand, has a component $M_{1,7}$ caused by the current in segment 1, and a component $M_{7,1}$ caused by the current in segment 7. The total mutual inductance linking these two segments equals

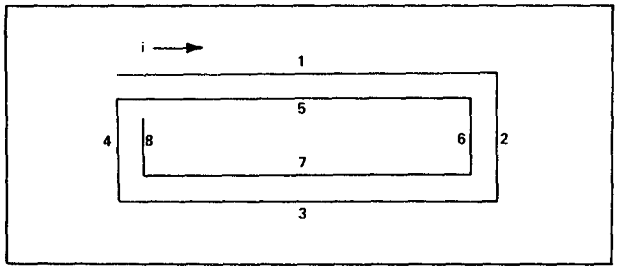

Fig. 3. Two-turn rectangular planar coil.

$M_{1,7}+M_{7,1}$ but is negative because current flow in segment 1 is opposite in direction to current flow in segment 7. An analogous relationship exists between segment pairs 1-3, 5-7, 5-3, 2-8, 2-4, 6-8, and 6-4. Current magnitude is identical in all segments, with the result that $M_{a,b}=M_{b,a}$. The total inductance L_T for this two-turn coil therefore becomes

$$L_T = L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 + 2(M_{1,5}+M_{2,6} + M_{3,7} + M_{4,8}) - 2(M_{1,7} + M_{1,3} + M_{5,7} + M_{5,3} + M_{2,8} + M_{2,4} + M_{6,8} + M_{6,4}).$$

The general equation for a coil or a part of a coil of any shape is

$$L_T = L_0 + \Sigma M \tag{8}$$

where L_T is the total inductance, L_0 is the sum of the self-inductances of all the straight segments, and ΣM is the sum of all the mutual inductances, both positive and negative. Since mutual inductance is positive when current flow in two parallel conductors is in the same direction and negative when current flow is in opposite directions, (8) can be rewritten to read

$$L_T = L_0 + M_+ - M_- \tag{9}$$

where M_+ is the sum of the positive mutual inductances and M_- is the sum of the negative mutual inductances.

The mutual inductance between two parallel conductors is a function of the length of the conductors and of the geometric mean distance between them. In general,

$$M = 2\ell Q \tag{10}$$

where M is the mutual inductance in nanohenries, ℓ is the conductor length in centimeters, and Q is the mutual-inductance parameter, calculated from the equation

$$Q = \ln \left\{ (\ell/\text{GMD})+[1+(\ell^2/\text{GMD}^2)]^{1/2} \right\} - [1+(\text{GMD}^2/\ell^2)]^{1/2} + (\text{GMD}/\ell). \tag{11}$$

In this equation, ℓ is the length corresponding to the subscript of Q, and GMD is the geometric mean distance between the two conductors, which is approximately equal to the distance d between the track centers. The exact value of the GMD may be calculated from the equation

$$\ln \text{GMD} = \ln d - \left\{[1/12(d/w)^2] + [1/60(d/w)^4] + [1/168(d/w)^6] + [1/360(d/w)^8] + [1/660(d/w)^{10}] + \cdots \right\} \tag{12}$$

where w is the track width.

Now consider the two-conductor geometry represented schematically in Fig. 4. Two filaments of lengths j and m, respectively, are separated by a geometric mean distance GMD. In this case,

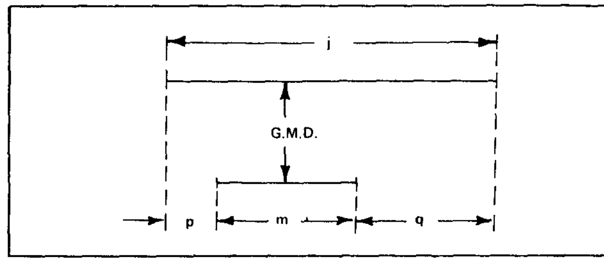

Fig. 4. Two-parallel-filament geometry.

$$2M_{j,m} = +(M_{m+p} + M_{m+q}) - (M_p + M_q) \tag{13}$$

and the individual M terms are calculated using equation (10) and the lengths corresponding to the subscripts; that is,

$$M_{m+p} = 2\ell_{m+p}Q_{m+p} = 2(m+p)Q_{m+p}$$

where Q_{m+p} is the mutual-inductance parameter Q for GMD/$(m+p)$. Though other more general expressions are available,[4] we will limit ourselves for purposes of this paper to the use of (13) and two additional relationships:

for $p = q$,

$$M_{j,m} = M_{m+p} - M_p \tag{14}$$

for $p = 0$,

$$2M_{j,m} = (M_j + M_m) - M_q. \tag{15}$$

SOME COMPARATIVE CALCULATIONS

In the sections that follow, we shall calculate by several methods the inductance of a single-turn square planar coil of the type shown in Fig. 5. All segments will be assumed to be shortened at each connecting end by half the track width w, so that

$$\ell_1 = \ell_2 = \ell_3 = 0.10 - w = 0.10 - 0.01 = 0.09 \text{ centimeter}$$

and

$$\ell_4 = \ell_2 - w - s = 0.09 - 0.01 - 0.01 = 0.070 \text{ centimeter}$$

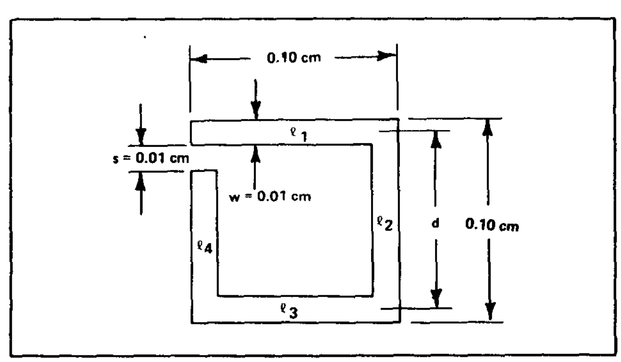

Fig. 5. Single-turn square planar coil.

[4]The derivation of (13) and these more general expressions are presented by Grover in [5].

We shall also assume that the magnetic permeability of the conductor material is 1 and that film thickness t is 0.0005 centimeter.

Expanded Grover Method

The derivations in the preceding theoretical discussion are based largely on the work of Grover [5] and produce the following calculated results. Repeating (9),

$$L_T = L_0 + M_+ - M_-$$

where $L_0 = L_1 + L_2 + L_3 + L_4$. From (7), we obtain

$$L_x = 2\ell_x \left\{ \ln[2\ell_x/(w+t)] + 0.50049 + [(w+t)/3\ell_x] \right\} \quad (16)$$

where L_x is the segment inductance in nanohenries, ℓ_x is the segment length in centimeters, w is the segment width in centimeters, and t is the segment thickness in centimeters. Substituting values into (16), we obtain

$$L_1 = 2(0.09) \left\{ \ln[(2)(0.09)/(0.01+0.0005)] + 0.50049 + [(0.01+0.0005)/(3)(0.09)] \right\}$$

or

$$L_1 = L_2 = L_3 = 0.60867 \text{ nanohenry.}$$

Similarly,

$$L_4 = 2(0.070) \left\{ \ln[(2)(0.070)/0.0105] + 0.50049 + [0.0105/(3)(0.070)] \right\} = 0.43597 \text{ nanohenry.}$$

Then,

$$L_0 = 3(0.60867) + 0.43597 = 2.26198 \text{ nanohenries.}$$

Since the currents in parallel legs flow in opposite directions, there is no positive mutual inductance in this coil; that is,

$$M_+ = 0$$

The negative mutual inductance is equal to the sum of $M_{1,3}$, $M_{3,1}$, $M_{2,4}$, and $M_{4,2}$, or, since $M_{1,3}$ equals $M_{3,1}$ and $M_{2,4}$ equals $M_{4,2}$,

$$M_- = 2(M_{1,3} + M_{2,4}). \quad (17)$$

Going back to (12) and substituting values of 0.01 and 0.09 for w and d, respectively, yields a GMD of 0.0899 centimeter. This value and that for ℓ_1, when substituted into (11), yield a mutual-inductance parameter Q_1 of 0.4672. Now, using (10) and the fact that ℓ_1 equals ℓ_3 we can write

$$M_{1,3} = 2\ell_1 Q_1 = 2(0.09)(0.4672) = 0.084096 \text{ nanohenry.}$$

However, because ℓ_2 does not equal ℓ_4, (15) must be used to solve for $M_{2,4}$. In this case,

$$2M_{2,4} = (M_2 + M_4) - M_{0.02}. \quad (18)$$

Again using (10),

$$M_2 = 2\ell_2 Q_2$$
$$M_4 = 2\ell_4 Q_4$$
$$M_{0.02} = 2(0.02)Q_{0.02}.$$

Since ℓ_2 equals ℓ_1 and the GMD remains constant, Q_2 must equal Q_1 as calculated from (11). It follows that

$$M_2 = M_{1,3} = 0.084096 \text{ nanohenry.}$$

To obtain Q_4 and $Q_{0.02}$, however, (11) must be solved for a GMD of 0.0899 and segment lengths of 0.070 and 0.02 centimeter, respectively. Thus calculated, Q_4 is found to be 0.3770, and M_4 becomes

$$M_4 = 2(0.070)(0.3770) = 0.052780 \text{ nanohenry.}$$

Similarly, $Q_{0.02}$ is found to be 0.0110, and $M_{0.02}$ becomes

$$M_{0.02} = 2(0.02)(0.011) = 0.000440 \text{ nanohenry.}$$

Substituting these values into (18), we obtain

$$2M_{2,4} = (0.084096 + 0.052780) - 0.000440$$
$$M_{2,4} = 0.068218 \text{ nanohenry.}$$

Having determined $M_{1,3}$ and $M_{2,4}$, we can now calculate the total negative mutual inductance in the coil, as expressed by (17):

$$M_- = 2(0.084096 + 0.068218) = 0.30463 \text{ nanohenry.}$$

Finally, returning to (9),

$$L_T = L_0 + M_+ - M_-$$

we obtain for total inductance

$$L_T = 2.26198 - 0.30463 = 1.9573 \text{ nanohenries.}$$

Note that, if we were to neglect the negative mutual inductance in the coil, L_T would equal L_0 and have a value of 2.26198 nanohenries. Alternatively, if the coil were treated as equivalent to a straight conductor equal in length to the sum of the segment lengths (0.335 centimeter), the inductance value arrived at would be

$$L = 2(0.335) \left\{ \ln[(2)(0.335)/0.0105] + 0.50049 + [0.0105/(3)(0.0335)] \right\} = 3.1479 \text{ nanohenries.}$$

Bryan Method

Bryan's equation for the inductance of a flat square coil [10], which has been referenced by Dukes [8] has the form

$$L = 0.141an^{5/3}\log[8(a/c)]$$

with dimensions expressed in inches and inductance in microhenries. In terms of centimeter dimensions and natural logarithms, the equation becomes

$$L = 0.0241an^{5/3}\ln[8(a/c)]$$

where a is outside plus inside diameter divided by 4, c is outside minus inside diameter divided by 2, and n is the number of turns.

Applying this equation to the one-turn coil represented in Fig. 5, for which

$$a = (0.10+0.08)/4 = 0.045 \text{ centimeter}$$

and

$$c = (0.10\text{-}0.08)/2 = 0.01 \text{ centimeter}$$

one obtains for total inductance

$$L = (0.0241)(0.045)(1) \left\{ \ln[8(0.045/0.01)] \right\} = 3.8871 \text{ nanohenries.}$$

Terman Method

Terman [7] has derived two inductance equations[5] that are applicable to the simple coil under consideration. One applies to a single-turn rectangle of rectangular wire and has the form

$$L = 0.02339 \left\{ (S_1+S_2)\log[2S_1S_2/(w+t)] - S_1\log(S_1+g) - S_2\log(S_2+g) + 0.01016 \left\{ 2g - [(S_1+S_2)/2] + 0.447(w+t) \right\} \right\}$$

where S_1 and S_2 are the maximum side lengths, g is the diagonal, w is the conductor width, and t is the conductor thickness, with dimensions expressed in inches and inductance in microhenries. For the case of a square, this equation becomes

$$L = (0.02339)(2S) \left\{ \log[2S^2/(w+t)] - \log(S+g) \right\} + 0.01016 \left\{ 2g - S + [0.447(w+t)] \right\}.$$

For the coil represented in Fig. 5,

$$S = 0.10 \text{ centimeter} = 0.0394 \text{ inch}$$
$$g = (1.414)(0.0394) = 0.0557 \text{ inch}$$
$$w + t = 0.0105 \text{ centimeter} = 0.00413 \text{ inch.}$$

Substituting these values into the equation above, one obtains

$$L = (0.02339)(2)(0.0394) \left\{ \log[2(0.0394)^2/0.00413] - \log(0.0394+0.0557) \right\} + 0.01016 \left\{ [2(0.0557)] - 0.0394 + [0.447(0.00413)] \right\}.$$

Then

$$L = [(1.844)(10^{-3})(-0.125 - 1.022)] + [(0.751)(10^{-3})] \text{ microhenries} = 2.403 \text{ nanohenries}.$$

Terman has also derived an equation for square coils of rectangular cross section that is good for any number of turns n. This equation,

$$L = 0.0467Sn^2 \left\{ \log[2S^2/(t+w)] - \log 2.414S \right\} + 0.02032Sn^2 \left\{ 0.914 + [0.2235(t+w)/S] \right\}$$

where dimensions are expressed in inches and inductance in microhenries, is simply a modification of the first and would yield identical results.

Other Methods

Inductance equations have been derived by Wheeler [11], Gleason [1], and Olivei [3], but they are limited to spiral geometries and cannot be applied to square or rectangular coils. The formula developed by Dill [2] for flat square geometry applies only to cases in which the coil area is completely filled.

Summary of Results

The inductances calculated for a single-turn coil by the various methods described above are summarized in Table II. The differences noted are particularly alarming when one considers that none of the methods used was derived for circular spirals and none assumed either a zero cross section or a circular cross section for the conductor. Indeed, all but the Bryan method took into consideration both the width and the thickness of the conductor. Though no direct measurements have been made on coils of the exact size represented in Fig. 5, measurements on other coils have been shown to agree with results calculated by the expanded Grover method within experimental error.[6]

As we have seen, the expanded Grover method is very lengthy and cumbersome, even for a single-turn coil. For a multiturn coil, calculations requiring as long as eight hours if performed without computer aid are not uncommon. The computer program described in the section that follows has proved an effective solution to this problem.

TABLE II
Comparison of Inductance Calculations for a Square Planar Single-Turn Coil

Calculation Method	Calculated Inductance, nanohenries
Expanded Grover Formula	1.9573
Grover Formula without Mutual Inductance	2.2620
Coil Considered a Straight Conductor	3.1479
Bryan Formula	3.8871
Terman Formula	2.403

COMPUTER PROGRAM FOR INDUCTANCE CALCULATIONS

Computer calculation of total inductance is based on (9) previously cited, namely,

$$L_T = L_0 + M_+ - M_-.$$

All straight segments of the induction coil are assigned serial numbers from 1 to Z, Z being the total number of segments. Numbering proceeds from outside to inside. Since Z need not be a multiple of 4, inductance can be calculated for coils with a resolution of a quarter turn. For a coil with four turns, Z equals 16; for a coil with 2¾ turns, Z equals 11. The data required for each calculation are the number of segments Z, the length of the first segment ℓ_1, the length of the second segment ℓ_2, the width of the conductor w, the thickness of the conductor t, the edge-to-edge distance between conductors s, and the number of complete turns n.

The computer calculates the lengths of all other segments. For even-numbered segments, it uses the expression

$$\ell_{2y} = \ell_2 - (y-1)(w+s) \tag{19}$$

and for odd-numbered segments,

$$\ell_{2y-1} = \ell_1 - (y-2)(w+s) \tag{20}$$

with $y \geqslant 2$. Then

[5]Equation (34) in [7] applies to a single-turn rectangle of rectangular wire; equation (60) applies to square coils of rectangular cross section.

[6]Supporting data are presented in a subsequent section.

$$L_0 = \sum_{y=1}^{Z} L_y \tag{21}$$

L_y being calculated using a form of (7), namely,

$$L_y = 0.002\ell_y \left\{ \ln[2\ell_y/(w+t)] + 0.50049 + [(w+t)/3\ell_y] \right\} \tag{22}$$

where inductance is in microhenries.

The number of terms contributing to M_+ increases rapidly with the number of segments in the coil. For n full turns and Z total segments, the number of positive mutual-inductance terms will be

$$4[n(n-1)] + 2n(Z-4n).$$

Since these terms have the general form

$$M_{y,(y+4n)}$$

the total positive mutual inductance may be represented

$$M_+ = \Sigma M_{y,(y+4n)} = 2[M_{y,(+4)}, M_{y,(y+8)}, M_{y,(y+12)}, \ldots] \tag{23}$$

where y has values from 1 through $Z-4$, n has values from 1 through the number of complete turns, and $y+4n$ has a maximum value of Z. Consider, for example, a coil having 3¼ turns, such as is diagrammed in Fig. 6. This coil, for which $n=3$ and $Z=13$, will have 30 positive mutual-inductance terms. The $M_{y,(y+4)}$ terms are $M_{1,5}$, $M_{2,6}$, $M_{3,7}$, $M_{4,8}$, $M_{5,9}$, $M_{6,10}$, $M_{7,11}$, $M_{8,12}$, and $M_{9,13}$. The $M_{y,(y+8)}$ terms are $M_{1,9}$, $M_{2,10}$, $M_{3,11}$, $M_{4,12}$, and $M_{5,13}$. The $M_{y,(y+12)}$ term is $M_{1,13}$. These 15 terms fall inside the bracket, so that the expression for total positive mutual inductance becomes

$$M_+ = 2[M_{1,5}+M_{2,6}+M_{3,7}+M_{4,8}+M_{5,9}+M_{6,10}+M_{7,11}+M_{8,12} +M_{9,13}+M_{1,9}+M_{2,10}+M_{3,11}+M_{4,12}+M_{5,13}+M_{1,13}].$$

Equation (14), which is used by the computer to calculate values for these individual terms, is an exact equation for all conductor pairs except those involving segment 1. As can be seen from Fig. 6, however, pairs of the latter type—in this case, 1-5, 1-9, and 1-13—are almost symmetrical, and using a symmetrical formula for them introduces only a very small error; moreover, since this error also exists in the calculation of negative mutual inductance, it tends to cancel out of the total inductance equation. Equation (14) can be rewritten in the form

$$M_{y,(y+4n)} = M_{(y+4n)+\{[y-(y+4n)]/2\}} - M_{\{[y-(y+4n)]/2\}}. \tag{24}$$

Then, combining (10) and (24), we obtain

$$M_{y,(y+4n)} = 2\ell_{(y+4n)+\{[y-(y+4n)]/2\}} Q_{(y+4n)+\{[y-(y+4n)]/2\}} - 2\ell_{\{[y-(y+4n)]/2\}} Q_{\{[y-(y+4n)]/2\}}. \tag{25}$$

The ℓ values are calculated using (19) and (20), and Q is calculated using (11).

Negative mutual inductance results from fluxes common to segments on opposite sides of the coil. The number of terms contributing to M_- is even greater than the number contributing to M_+. For a coil having n full turns and Z total segments, it equals

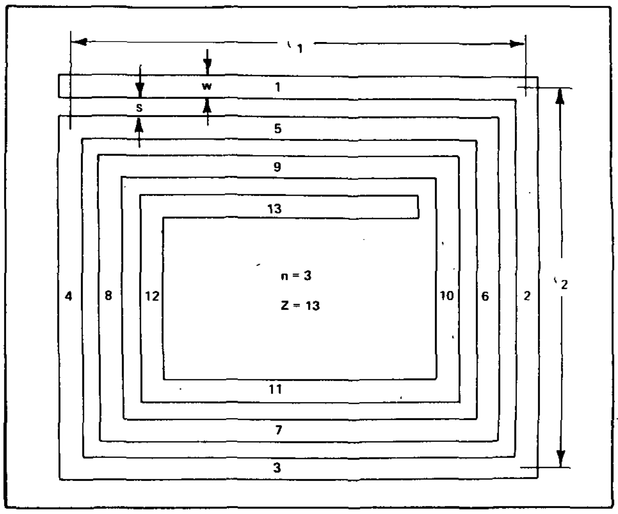

Fig. 6. Square planar 3¼-turn coil.

$$4n^2 + 2n(Z-4n) + (Z-4n-2)(Z-4n-1)[(Z-4n)/3].$$

Negative mutual-inductance terms have the general form

$$M_{y,(y+4n-2)}$$

and total negative mutual inductance may be represented

$$M_- = \Sigma M_{y,(y+4n-2)} = 2[M_{y,(y+2)}, M_{y,(y+6)}, M_{y,(y+10)}, \ldots] \tag{26}$$

where y has values from 1 through $Z-2$, n has values from 1 through the number of complete turns, and $y+4n-2$ has a maximum value of Z. The coil in Fig. 6, for which $n=3$ and $Z=13$, will have a total of 42 negative mutual-inductance terms. The $M_{y,(y+2)}$ terms are $M_{1,3}$, $M_{2,4}$, $M_{3,5}$, $M_{4,6}$, $M_{5,7}$, $M_{6,8}$, $M_{7,9}$, $M_{8,10}$, $M_{9,11}$, $M_{10,12}$, and $M_{11,13}$. The $M_{y,(Y+6)}$ terms are $M_{1,7}$, $M_{2,8}$, $M_{3,9}$, $M_{4,10}$, $M_{5,11}$, $M_{6,12}$, and $M_{7,13}$. The $M_{y,(y+10)}$ terms are $M_{1,11}$, $M_{2,12}$, and $M_{3,13}$. These 21 terms fall inside the bracket, so that the expression for total negative mutual inductance becomes

$$M_- = 2[M_{1,3}+M_{2,4}+M_{3,5}+M_{4,6}+M_{5,7}+M_{6,8}+M_{7,9} +M_{8,10}+M_{9,11}+M_{10,12}+M_{11,13}+M_{1,7}+M_{2,8}+M_{3,9} +M_{4,10}+M_{5,11}+M_{6,12}+M_{7,13}+M_{1,11}+M_{2,12}+M_{3,13}].$$

Values for these negative mutual-inductance terms are calculated in much the same manner as those for positive mutual inductance. Rewritten for this calculation, (14) takes the form

$$M_{y,(y+4n-2)} = M_{(y+4n-2)+\{[y-(y+4n-2)]/2\}} - M_{\{[y-(y+4n-2)]/2\}} \tag{27}$$

Then, combining (10) and (27), we obtain

$$M_{y,(y+4n-2)} = 2\ell_{(y+4n-2)+\{[y-(y+4n-2)]/2\}} Q_{(y+4n-2)+\{[y-(y+4n-2)]/2\}} - 2\ell_{\{[y-(y+4n-2)]/2\}} Q_{\{[y-(y+4n2)]/2\}}. \tag{28}$$

As in the case of the positive terms, ℓ values are calculated using (19) and (20) and Q is calculated using (11).

COMPUTER-CALCULATED INDUCTANCE DATA

Four different induction coils were fabricated by vacuum-depositing phased chromium/gold onto 99 percent alumina and then gold-plating to a thickness of 0.0005 inch (0.0127 millimeter). Values for L and Q were measured at 150 megahertz, and the corresponding L values were calculated by the Bryan, Terman, and expanded Grover methods. Results are summarized in Table III. It will be noted that Terman's method yielded values two to four times higher than the measured values. Bryan's method proved better, though the results for small coils were much too high.

Figs. 7 through 12 present plots developed from other computer-calculated inductance data for square and rectangular planar coils.[7] In Figs. 7, 9, and 11, inductance in nanohenries is plotted versus number of segments Z for various lengths ℓ_1 of the first outside segment, the number of complete turns n being equal to $Z/4$. In Figs. 8, 10, and 12, inductance in nanohenries is plotted versus segment length ℓ_1 for different numbers of complete turns n.

The irregularities in the Fig. 9 and Fig. 11 curves result from the fact that segment lengths between the outside and

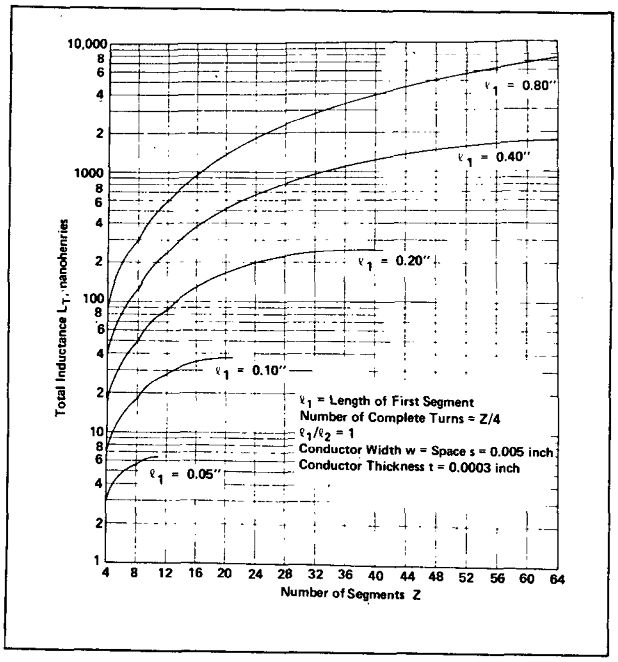

Fig. 7. Square-planar-coil inductance as a function of number of coil segments.

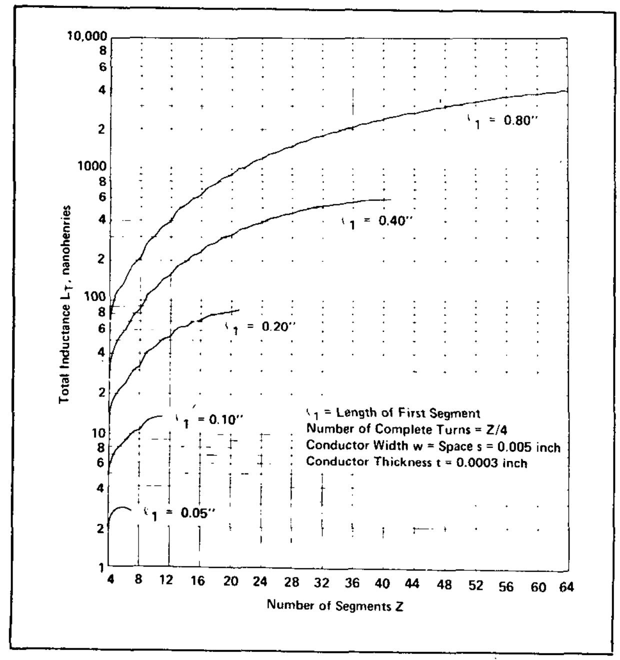

Fig. 9. Rectangular-planar-coil inductance as a function of number of coil segments ($\ell_1/\ell_2 = 2$).

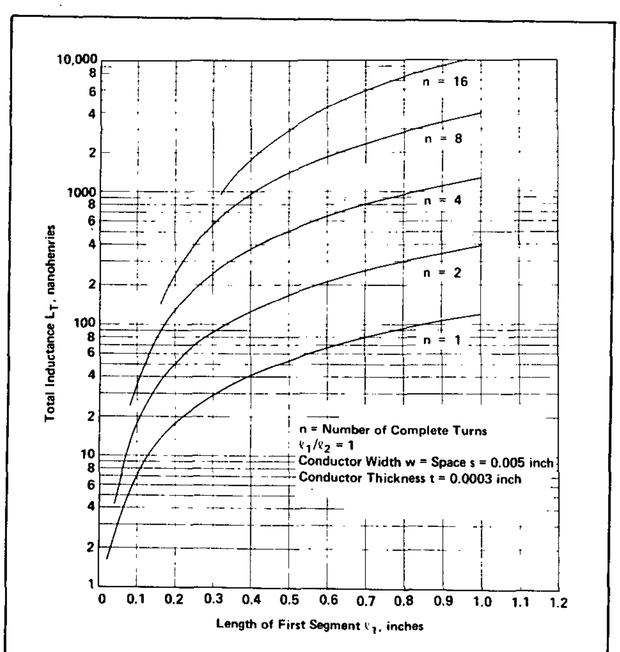

Fig. 8. Square-planar-coil inductance as a function of first-segment length.

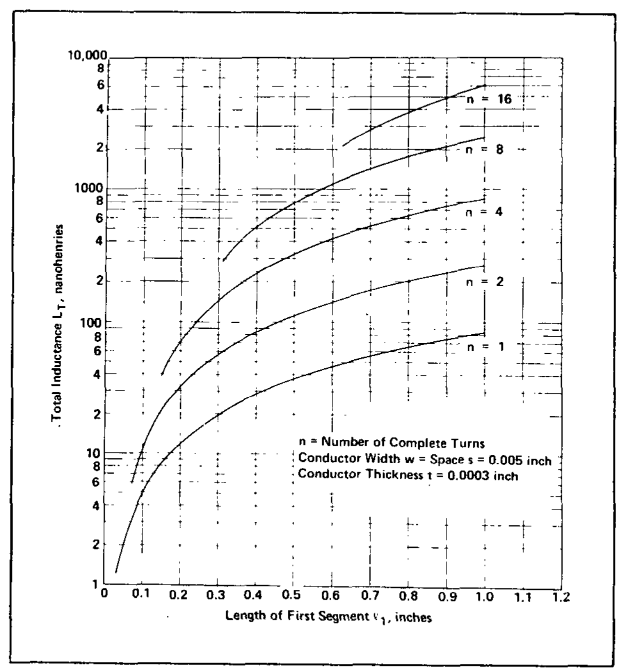

Fig. 10. Rectangular-planar-coil inductance as a function of first-segment length ($\ell_1/\ell_2 = 2$).

[7]Exploded copies of these figures with much finer resolution are available from the author upon request.

the inside of the coil do not decrease uniformly. Segments that are longer than the preceding segments contribute greater amounts of inductance and cause fluctuations in the value of L_T. Since the amount of this fluctuation constitutes a decreasing percentage of total inductance as the number of segments increases, the irregularities become less pronounced as the semilogarithmic plot progresses.

The slight irregularities in the Fig. 7 curves result from a similar phenomenon. The fact that ℓ_1 has been defined as

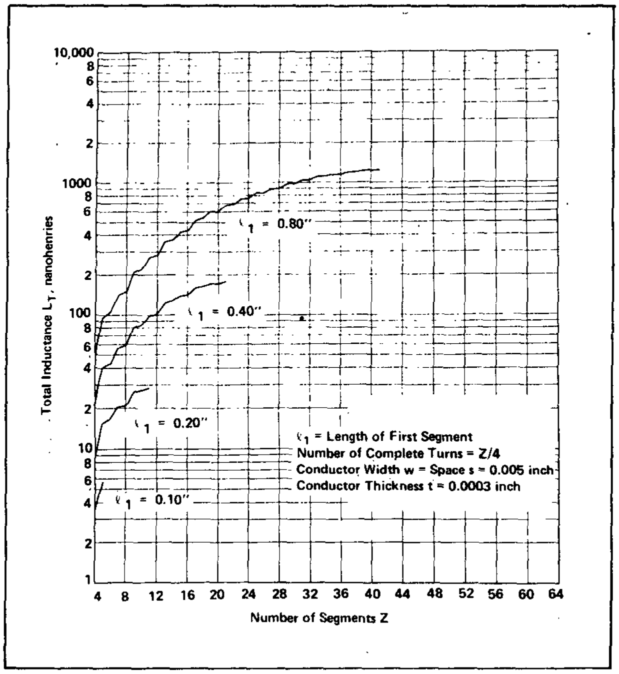

Fig. 11. Rectangular-planar-coil inductance as a function of number of coil segments ($\ell_1/\ell_2 = 4$).

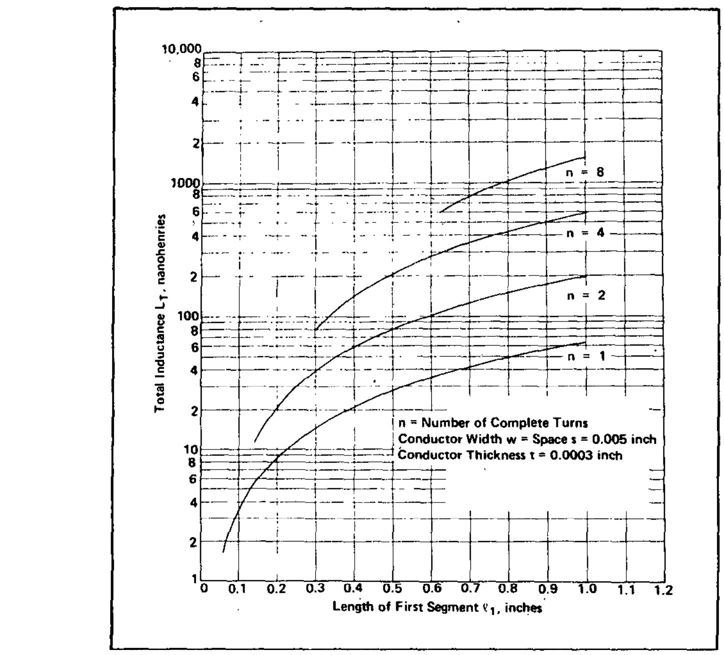

Fig. 12. Rectangular-planar-coil inductance as a function of first-segment length ($\ell_1/\ell_2 = 4$).

TABLE III
Calculated as Compared with Measured Inductance Values for Typical Square Coils

Method of Determination	Inductance, nanohenries			
	Coil A	Coil B	Coil C	Coil D
Bryan Calculation	35.6	71.4	111.4	207.3
Terman Calculation	67.13	111.7	447.6	636.2
Expanded Grover Calculation	28.33	56.84	106.68	197.88
Experimental Measurement	23.2	51.8	98.9	211.9

equal to ℓ_2 in a square coil causes an initial nonuniformity in the rate of decrease of ℓ, the pattern being

$$\ell_1 = \ell_2 = \ell_3 > \ell_4 = \ell_5 > \ell_6 = \ell_7 > \ell_8 = \ell_9 > \ell_{10} \cdots$$

Moreover, the addition of new segments does not add uniformly to the number of the positive and negative mutual-inductance terms. The addition of segments 5, 6, 9, 10, 13, 14, etc. introduces two positive and two negative terms per segment whereas with the addition of segments 7, 8, 11, 12, 15, 16, etc., the number of negative terms introduced per segment exceeds the number of positive terms by two.

Two important types of information can be readily extracted from Figs. 7 through 12. It is possible to determine by inspection not only the inductance value that corresponds to a given coil geometry, but also the various coil geometries that will yield a given inductance value. Information of the latter type is particularly valuable and is difficult to obtain in a practical manner without the aid of a computer. Even assuming that an accurate method of calculating inductance for a given geometry is available, the search for a geometry that will yield a given inductance is inevitably an iterative process that begins with a best guess. The calculation must usually be repeated a number of times before it is possible to single out a geometry the inductance value for which is close enough to permit fabrication of the coil. Using Figs. 7 through 12, however, the task becomes relatively simple.

Assume, for example, that an inductance of 100 nanohenries is required. Fig. 7 indicates that this inductance value can be obtained using a square coil of just over four segments with an ℓ_1 of 0.80 inch (2.032 centimeters), a square coil of almost seven segments with an ℓ_1 of 0.40 inch (1.016 centimeters), and a square coil of 13 segments with an ℓ_1 of 0.20 inch (0.508 centimeter). Fig. 8 indicates that the same inductance can be obtained using a square two-turn coil with an ℓ_1 of 0.34 inch (0.864 centimeter) or a square one-turn coil with an ℓ_1 of 0.85 inch (2.160 centimeters). Examining each of the other figures in the same manner, one finds that all the geometries described in Table IV will, in fact, yield the desired inductance value. Such a tabulation provides a great deal of latitude in overall hybrid microcircuit design. Although Q values for these coils have not been thoroughly investigated, we have observed that they range typically between 20 and 45, provided that the thickness of the inductor is adequate for the frequency stipulated. A thickness of 0.0003 inch (0.0076 millimeter) is required at 3 gigahertz and a thickness of 0.0001 inch (0.025 millimeter) at 10 megahertz.

TABLE IV

Rectangular-Planar-Coil Geometries for a 100 Nanohenry Inductor

Coil Parameters			
ℓ_1/ℓ_2	ℓ_1, inches (centimeters)	Number of Segments Z	Number of Turns n
1	0.80 (2.032)	4+	
1	0.40 (1.016)	7−	
1	0.20 (0.508)	13	
1	0.17 (0.432)	16	4
1	0.34 (0.864)	8	2
1	0.85 (2.159)	4	1
2	0.80 (2.032)	4½	
2	0.40 (1.016)	8½	
2	0.20 (0.508)	16	4
2	0.46 (1.168)	8	2
4	0.80 (2.032)	5½	
4	0.40 (1.016)	11½	
4	0.33 (0.838)	16	4
4	0.59 (1.499)	8	2

CONCLUSION

The concept of negative mutual inductance has been discussed, and equations for calculating mutual inductance as well as total inductance for planar rectangular coils have been presented. A computer program designed to solve these equations has been described, and the utility of its graphic data output for coil design has been demonstrated. Sufficient detail has been presented to permit the interested reader to develop similar computer programs for other inductor types.

SYMBOLS

- AMD Arithmetic mean distance.
- d Distance between track centers.
- g Diagonal of coil cross section.
- GMD Geometric mean distance.
- i Current.
- ℓ Conductor length.
- L Self-inductance.
- L_0 Sum of self-inductances (total minus mutual inductance).
- L_T Total inductance.
- L_x Self-inductance of coil segment x.
- M Mutual inductance.
- $M_{a,b}$ Mutual inductance between segments a and b due to $\phi_{a,b}$.
- M_T Total mutual inductance.
- M_+ Sum of positive mutual inductances.
- M_- Sum of negative mutual inductances.
- n Number of complete turns in coil.
- Q Mutual-inductance parameter.
- r Radius of conductor cross section.
- s Edge-to-edge distance between conductors.
- S Maximum side length.
- t Conductor thickness.
- T Frequency-correction parameter.
- w Conductor width.
- Z Total number of coil segments.
- μ Conductor permeability.
- $\phi_{a,b}$ Magnetic flux common to segments a and b and generated in a.

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