Calculation of Leakage Inductance in Transformer Windings

William G. Hurley, *Senior Member, IEEE*, and David J. Wilcox, *Member, IEEE*

***Abstract*—A formula is presented to calculate mutual impedance between transformer windings on ferromagnetic cores. The formula is based on the solution of Maxwell's Equations for coils on ferromagnetic cores and as such offers the ultimate in accuracy. The formula is frequency dependent, taking into account the effect of eddy currents in the core on the flux distribution as well as representing the eddy current core loss as an equivalent resistance. Experimental results are presented for leakage inductance and an illustrative example is presented showing how leakage inductance affects the operation of a typical switching mode power supply. Approximations for the formula are also presented to simplify the calculations under certain operating conditions.**

Nomenclature

In general, subscript 1 refers to coil 1 and subscript 2 refers to coil 2.

A	Average coil radius.
A_c	Core cross-sectional area.
a_1, a_2	Inside and outside radii of a coil.
b	Core radius.
GMD	Geometric mean distance between coils.
h	Radial height of coil.
$I_n(x), K_n(x)$	Modified Bessel functions of the first and second kind, respectively.
L_1	Self inductance associated with primary (air) field.
L_{11}	Self inductance of a coil.
L_{21}	Component of L_{11} due to flux linking coil 2.
L_c	Self inductance associated with reflected field.
L_l	Leakage inductance.
$\mathbf{L}_v$	Struve Function.
l	Magnetic path length of core (measured on center axis).
M_1	Mutual inductance associated with primary (air) field.
m	$\sqrt{(j\omega\mu_0\sigma_{\text{cu}})}$.
N	Number of turns in coil.
R	Average coil radius.
r_1, r_2	Inside and outside radii of a coil.
w	Width of coil.
Z	$R + j\omega M$, Mutual impedance
Z_c	$R_c + j\omega M_c$, Mutual impedance associated with reflected field.
Z_i	Internal impedance of a conductor.
z	Distance between filaments or coil centers, measured along the core axis.
Γ	$\sqrt{(\beta^2 + j\omega\mu_z\mu_0\sigma_{\text{co}})}$
ϕ_{12}	Flux linking coils 1 and 2 due to current in coil 2.
μ_0	Magnetic permeability of free space. $4\pi \times 10^{-7}$ H/m.
μ_z	Relative permeability of core.
σ_{co}	Conductivity of the core material.
σ_{cu}	Conductivity of the winding conductor.
ω	Angular frequency.

Manuscript received July 30, 1990; revised September 8, 1993. This work was partially funded by EOLAS, the Irish Science and Technology Agency, under the program of advance technology in Power Electronics.

The authors are with the Department of Electronic Engineering, University College, Galway, Ireland.

IEEE Log Number 9214838.

I. Introduction

IT IS A well-known fact that successful miniaturization of power supplies requires that magnetic components, such as transformers and inductors, operate at higher frequencies. Manufacturers depend on empirical models to predict high frequency operation of ferromagnetic cores. Power densities of 100 W/in^3 are only possible, with current materials, by switching at frequencies approaching 10 MHz. The inductance of a coil on a ferrite core at 10 MHz is quite different from the inductance of the same coil at 100 kHz. Clearly a precise frequency dependent model is required to investigate the limits of current materials and to characterize those properties which make high frequency operation possible. The issues involved in transformer design at 10 MHz are clearly enunciated in [1].

Previous work [2] has shown that it is possible to take proper account of frequency and eddy current effects in transformer and inductor cores by means of a new frequency dependent impedance formula. Validation tests [3] have shown excellent agreement between predicted and measured results. While previous work concentrated on resonant effects in large power transformers, the present contribution concerns itself with toroidal cores used in Switched-Mode Power Supplies (SMPS). Leakage inductance plays a very important role in Pulse Width Modulated (PWM) converters, limiting the upper frequency of operation. More recent developments in resonant power supplies depend on precise knowledge of leakage inductance for correct design and operation. In this paper the new formula [2] is used for leakage inductance calculation in toroidal cores and various configurations are examined. A typical PWM forward converter is analyzed and the effects of leakage inductance on circuit operation are explored. Test results for leakage inductance are presented and compared with predictions from the model.

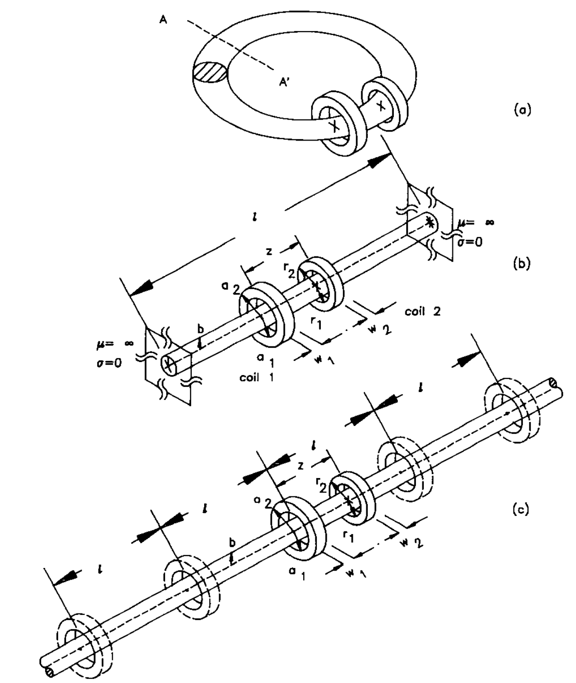


Fig. 1. (a) Toroidal core. (b) Equivalent layout. (c) Images.

Several authors have addressed the leakage inductance problem [4]-[6]. In this paper, leakage inductance is presented as part of a more general formula for the full mutual impedance between windings on a ferromagnetic core and should therefore shed additional light on the leakage inductance problem, particularly at high frequencies.

II. Mutual Impedance in a Toroidal Core

The detailed derivation of the formula is given in [2] and is summarised here for completeness. The closed magnetic path of the toroidal core, as shown in Fig. 1(a), is modeled by imagining the actual core to have been cut open at A A' and straightened out to its length l. The magnetic circuit is then reclosed by providing a return flux path of zero reluctance between the two ends. This is done by placing the straightened out core between two infinite plates of ideal magnetic material ($\mu = \infty, \sigma = 0$) as shown in Fig. 1(b). Maxwell's Equations are solved for the structure of Fig. 1(b) using integral transform techniques, as described in [2]. Experimental results [3] have shown that the structure of Fig. 1(b) is a very good approximation to the closed core. Since, in practice, the length of the core is much greater than its diameter, the field in the vicinity of the coils is not unduly affected.

Furthermore, it can be shown that the magnetic endplates in Fig. 1(b) can be removed and replaced by images of coil 1 at distances $z = \pm kl$, $k = 1, 2, 3 \ldots$ on a core of infinite length as shown in Fig. 1(c). This is a mathematical device to satisfy the field boundary conditions in the case of a closed core [2].

The mutual impedance between the two coils is given by

$$Z = j\omega M_1 + Z_c \tag{1}$$

$$Z_c = j\omega\mu_0 \frac{N_1 N_2}{h_1 w_1 h_2 w_2} \frac{2\pi}{l} \left\{ \frac{h_1 w_1 h_2 w_2 b^2}{2} \left(\frac{2\mu_z I_1(\Gamma_0 b)}{\Gamma_0 b I_0(\Gamma_0 b)} - 1 \right) + 2\sum_{k=1}^{\infty} P(\beta_k a_2, \beta_k a_1) P(\beta_k r_2, \beta_k r_1) Q(\beta_k w_1, \beta_k w_2) \times \Phi(\beta_k) \cos(\beta_k z) \right\} \tag{2}$$

$$\Phi(\beta) = \frac{I_0(\beta b)}{K_0(\beta b)} \left\{ \frac{1 - \frac{I_1(\beta b)\Gamma b I_0(\Gamma b)}{\mu_z \beta b I_0(\beta b) I_1(\Gamma b)}}{1 + \frac{K_1(\beta b)\Gamma b I_0(\Gamma b)}{\mu_z \beta b K_0(\beta b) I_1(\Gamma b)}} \right\} \tag{3}$$

and

$$\begin{aligned} \Gamma &= \sqrt{(\beta^2 + j\omega\mu_z\mu_0\sigma_{co})} \\ \beta_k &= k2\pi/l \\ \Gamma_0 &= \sqrt{(j\omega\mu_z\mu_0\sigma_{co})} \\ P(\beta x, \beta y) &= [p(\beta x) - p(\beta y)]/\beta^2 \\ p(z) &= \pi z[K_1(z)\boldsymbol{L}_0(z) + \boldsymbol{L}_1(z)K_0(z)]/2 \\ Q(\beta x, \beta y) &= 2\{\cos[\beta(x-y)/2] - \cos[\beta(x+y)/2]\}/\beta^2 \end{aligned}$$

M_1 is the mutual inductance between coil 2 and, coil 1 and its images as shown in Fig. 1(c), without the core. These mutual inductances can be calculated from tables given by Grover [7]. Alternatively the coils may be replaced by equivalent filaments separated by an axial distance equal to the geometric mean distance (GMD) between the coils or in the case of self inductance, the geometric mean distance of the coil cross-section. In this approximation, M_1 is given by:

$$M_1 = \mu_0 N_1 N_2 AR \frac{2\pi}{l} \left\{ \frac{R}{2A} + 2\sum_{k=1}^{\infty} I_1(\beta_k R) K_1(\beta_k A) \cos(\beta_k \mathrm{GMD}) \right\} \tag{4}$$

A is the mean radius of coil 1 and R is the mean radius of coil 2. The remaining dimensions are given in Fig. 1.

Z_c represents the reflected field due to the presence of the core. Z_c has both real and imaginary components ($Z_c = R_c + j\omega M_c$). R_c represents the eddy current loss in the core and M_c represents the additional coil inductance due to the core presence.

I_n, K_n are modified Bessel functions of the first and second kind respectively, $\boldsymbol{L}_v$ is the modified Struve function [8].

Note that there are there are two components in (2) and (4). The first component is independent of the distance z between the coils and corresponds to the straight lines of flux which link all the coils on the core. The summation terms represent the effects of flux which is outside the core, including flux which escapes from the core. In short, the summation terms represent leakage flux.

A. Self Impedance Tests

Self impedance tests were carried out on a Micrometals T400-26 10 cm O.D. powdered core with a distributed gap.

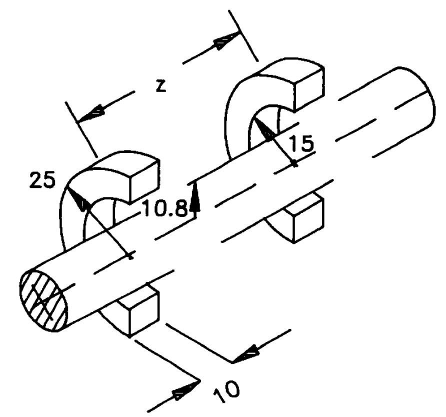


Fig. 2. Test coil dimensions (mm).

Full details of the test core are given in the Appendix, and the coil dimensions are given in Fig. 2. Measurements were taken using a Hewlett Packard impedance analyzer, from 100 Hz to 1 MHz. Calculation of self impedance is based on (1)–(4), ($z = 0$), and both measured and calculated results are shown in Fig. 3. Note that measurements beyond about 1 MHz are not practical as a consequence of the inter-turn capacitance effects in the coil. The value of core relative permeability given by the manufacturer is 75 which is low on account of the distributed gap. The value of effective core resistivity for the best fit is 0.01 Ω-m, which is much higher than the intrinsic resistivity of the iron on account of eddy current suppression inherent in powdered iron design.

It is interesting to compare the above results with the classical formula for inductance of toroidal cores. At low frequencies $\Gamma_0 = \sqrt{(j\omega\mu_z\mu_o\sigma_{co})} \rightarrow 0$ and from [8]

$$\lim_{\Gamma_0 \rightarrow 0} \frac{\Gamma_0 b I_0(\Gamma_0 b)}{I_1(\Gamma_0 b)} \rightarrow 2$$

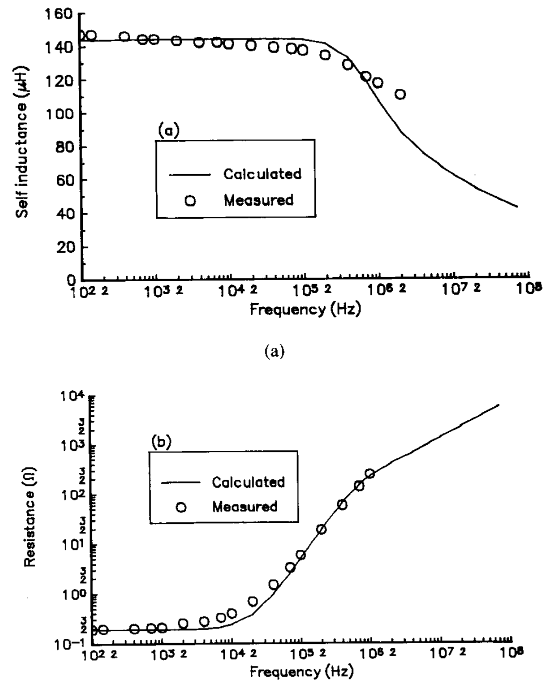
and if $\mu_z \gg 1$ then the inductance due to the constant core term becomes, for a simple coil of N turns,

$$L_c = \frac{\mu_z\mu_0 N^2 \pi b^2}{l} = \frac{\mu_z\mu_0 N^2 A_c}{l} \tag{5}$$

which is recognized as the classical formula for the inductance of a toroid. Further insight into the nature of leakage inductance can be obtained by examining the conditions which reduce the proposed formula to the familiar, and widely used, low frequency formula:

- Neglect air term.
- Core taken to be lossless with high magnetic permeability.
- Flux density uniform over the core cross-section.
- Ignore terms corresponding to leakage flux.

The self inductance of the test coil, calculated from the classical formula (5), is 85 μH (manufacturer's data gives 81μH). The calculated value using the new formula (1), which



(a)

(b)

Fig. 3. (a) Inductive component of aelf impedance. (b) Resistive component of aelf impedance.

agrees with the measured value, is 144 μH at 100 Hz. This is made up as follows:

constant core term	85 μH
summation core term	34 μH
constant air term	8 μH
summation air term	17 μH

The classical formula predicts the constant core term as expected, but the other components are almost equally important, making up 41% of the total.

In the case of a coil which is wound on the core so that w, the coil width, is equal to the core length l, then $Q(\beta x, \beta y)$ in the summation term of (2) becomes

$$\frac{4\sin^2(\beta_k l/2)}{\beta_k^2} = \frac{4\sin^2(2\pi k l/2l)}{\beta_k^2} = 0$$

causing the summation term to disappear. In this case there is no leakage flux and the inductance reduces to that given by the classical toroidal formula. This is what the manufacturer's data is based on. Clearly, in a practical situation where there are many windings on a core the simplified formula is quite inadequate.

The calculations for Fig. 3 include a correction for the internal impedance of the coil conductor, due to the skin effect, given by [9]

$$Z_i = R_{dc}\frac{mrI_0(mr)}{2I_1(mr)}$$
$$m = \sqrt{(j\omega\mu_0\sigma_{cu})}$$

where σ_{cu} is the conductivity of the copper conductor and r is the conductor radius. In Fig. 3(b) the value of resistance at low frequency is that of the coil and test leads (0.19 Ω). At the lower frequencies, say up to 50 kHz, self inductance is high and self resistance is low. However at the higher frequencies there is a substantial reduction in inductance and a substantial increase in losses. The deterioration in performance, accurately predicted by the formula, explains why the manufacturers deem the core unsuitable for operation above 50 kHz.

B. Approximations for the Impedance Formula

It is unfortunate that the proposed formula is so complicated and it is natural to seek simplified working approximations to it. There are two regions of the self inductance curve of Fig. 3(a) which show asymptotic behaviour, one at low frequency and the other at high frequency. The formulas for Γ and Γ_0 show that frequency and conductivity appear together as the product $\omega\sigma$. In this context low frequency and low conductivity are interchangeable. A core exhibiting low conductivity, such as a ferrite, would have a flat self inductance characteristic up to about 1 MHz. This justifies a simplification of (3). Noting that in practice $l/b > 25$ for toroidal cores and $\mu_z \gg 1$,

$$\Phi(\beta) \approx \frac{I_0(\beta b)}{K_0(\beta b)}$$

The constant core term is given by the classical toroidal formula, (5), since $\Gamma_0 \rightarrow 0$. As expected the result gives inductance only and the reflected component L_c of the self inductance of a coil of N turns is given by:

$$L_c = \frac{\mu_z \mu_0 N^2 A_c}{l} + \frac{\mu_0 N^2 2\pi}{(hw)^2 l} \times 2\sum_{k=1}^{\infty} P^2(\beta_k a_2, \beta_k a_1) Q(\beta_k w, \beta_k w)\Phi(\beta_k) \quad (6)$$

The high frequency asymptote of the self inductance curve in Fig. 3(a) corresponds to very high frequency or very high conductivity and represents the limiting case of zero flux penetration into the core (as a result of skin effect caused by eddy currents). In this case $\Gamma \rightarrow \infty$ and

$$\lim_{\Gamma_0 \rightarrow \infty} \frac{I_1(\Gamma_0 b)}{\Gamma_0 b I_0(\Gamma_0 b)} \rightarrow 0$$

The self inductance of a coil with N turns is then:

$$L = L_1 - \frac{\mu_0 N^2 A_c}{l} + \frac{\mu_0 N^2 2\pi}{(hw)^2 l} \times 2\sum_{k=1}^{\infty} P^2(\beta_k a_2, \beta_k a_1) Q(\beta_k w, \beta_k w)\Phi(\beta_k) \quad (7)$$

and

$$\Phi(\beta) = -\frac{I_1(\beta b)}{K_1(\beta b)}$$

The inductance in this case is actually reduced by the presence of the core. Evidently it is independent of the magnetic permeability of the core.

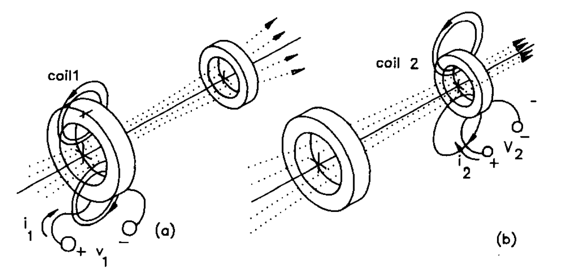


Fig. 4. Leakage inductance.

III. Leakage Inductance

Leakage inductance is the property of one winding with respect to another. Consider two air coils shown in Fig. 4. If the instantaneous current in coil 1 is i_1 and coil 2 is open circuited, the general character of the magnetic field produced by i_1 is shown in the field map of Fig. 4(a). The dotted lines show the flux linking both coils and the solid lines indicate flux which links coil 1 only, this is leakage flux of coil 1. Fig. 4(b) shows the corresponding situation when coil 2 is carrying current i_2 and coil 1 is open circuited. The nature of the two leakage fields is quite different. With both coils energized the flux in each coil has three components,

$$\phi_{11} = \phi_{l1} + \phi_{21} + \phi_{12}$$
$$\phi_{22} = \phi_{l2} + \phi_{12} + \phi_{21}$$

where ϕ_{l1} is the leakage flux due to i_1, ϕ_{21} is the flux linking both coils due to i_1, ϕ_{12} is the flux linking both coils due to i_2 and ϕ_{l2} is the leakage flux due to i_2. In terms of inductance we can write the voltages at the terminals of each coil,

$$V_1 = [L_{l1} + L_{21}]\frac{di_1}{dt} + M_{12}\frac{di_2}{dt}$$
$$V_2 = M_{21}\frac{di_1}{dt} + [L_{l2} + L_{12}]\frac{di_2}{dt}$$

The self inductance of coil 1 is L_{11},

$$L_{11} = L_{l1} + L_{21}$$

which can be measured. We can then extract the leakage term by noting that:

$$L_{21} = (N_1/N_2)M_{21}$$
$$L_{12} = (N_2/N_1)M_{12}$$

$M_{12} = M_{21} = M$ is the mutual inductance between the two coils.

$$L_{l1} = L_{11} - (N_1/N_2)M \quad (8)$$

M in this case is simply the inductive component of Z in (1) and L_{11} is found from (1) with $z = 0$.

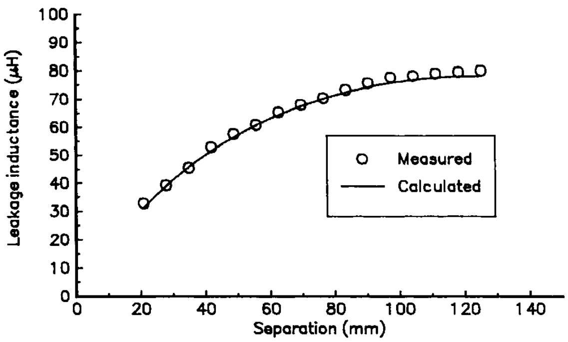


Fig. 5. Calculated and measured leakage inductance.

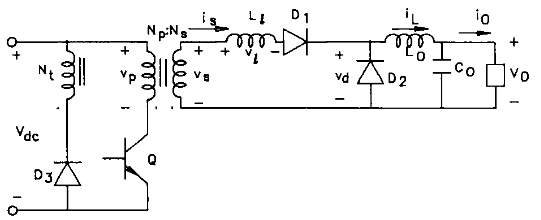


Fig. 6. PWM forward converter.

C. Leakage Inductance Tests

Leakage inductance tests were carried out on the 10 cm O.D. powdered iron core. Two identical coils were used and full details are given in the Appendix. The calculated leakage inductance is found from (8). The results are summarised in Fig. 5; clearly agreement is very good. These results were obtained at 1 kHz. At $z = 60$ mm, the leakage inductance is 63 μH. This consists of 24.6 μH for the air term and 38.4 μH for the core term. Many approximations for leakage inductance assume that all the leakage flux is in the air only, clearly these results show that this need not be the case.

IV. Leakage Inductance in a PWM Converter

Fig. 6 shows a forward converter with a reset winding. When the switch Q is closed, the input DC voltage is applied across the primary winding of the transformer and the flux builds up in the core as shown in Fig. 7(c). During this time diode D_1 is forward biased and diode D_2 is reverse biased. When switch Q is opened, the flux in the core is reset through diode D_3. Under ideal conditions, where there is no leakage inductance in the transformer, diode D_1 is now reverse biased and diode D_2 is forward biased. The detailed operation of this circuit is given in [10].

Ideally, current should commutate instantaneously from D_1 to D_2 and vice versa. When leakage inductance is present in the transformer secondary, instantaneous commutation is not possible. When Q is turned on the current in D_1 builds up through the leakage inductance L_l and diode D_2 remains on until the load current is established in D_1. Since both diodes are on, the secondary voltage appears across the leakage inductance

$$\frac{N_s}{N_p} V_{\mathrm{dc}} = L_l \frac{I_0}{\tau_{\mu 1}}$$

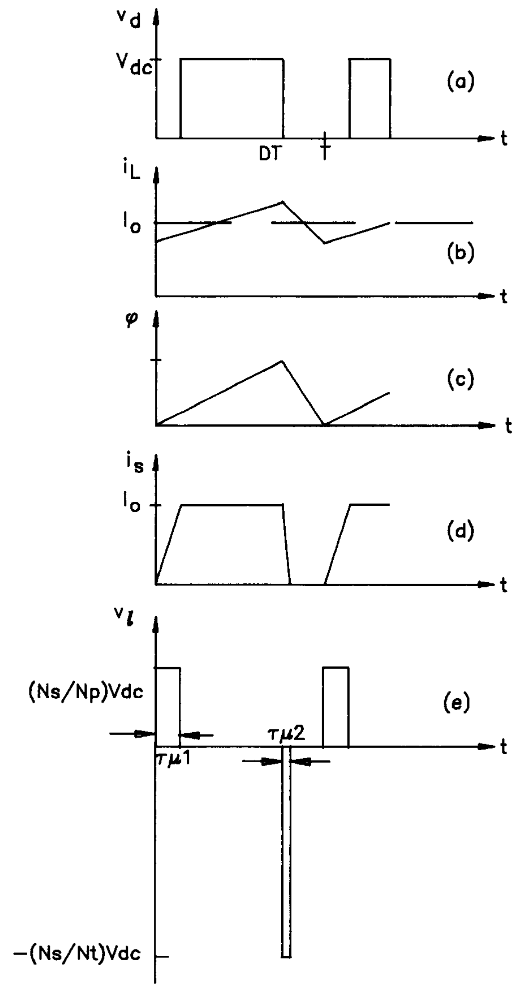


Fig. 7. Voltage and Current Waveforms ($N_s = N_p = 3N_t$).

giving

$$\tau_{\mu 1} = \frac{N_p L_l I_0}{N_s V_{\mathrm{dc}}}$$

We are assuming that the ripple in the output current is small as it should be in a well designed converter.

When the load current is established in D_1, D_2 is turned off.

When Q is turned off, the current in D_1 decreases linearly to zero and the secondary voltage across the leakage inductance is $-(N_s/N_t)V_{\mathrm{dc}}$ so that the turn-off commutation period is

$$\tau_{\mu 2} = \frac{N_t L_l I_0}{N_s V_{\mathrm{dc}}}$$

Since the input voltage is not presented to the output during the turn on commutation period, the average output is reduced from its ideal value

$$V_0 = [D - \tau_{\mu 1}/T] V_{\mathrm{dc}}$$

where D is the duty cycle and T the period of the converter.

Another consequence of the leakage inductance is that the maximum frequency of operation is limited by $\tau_{\mu 1} + \tau_{\mu 2}$. This has serious implications for PWM control, because higher frequencies lead to smaller magnetic components and higher power densities.

V. Conclusion

A formula has been presented which allows accurate calculation of leakage inductance in a transformer. The formula has been experimentally validated for a powdered iron core. It has been shown that the classical formula can be grossly inadequate for self inductance calculations. It has also been shown that neglecting the core in leakage inductance calculations can lead to inaccurate results.

It has been shown that leakage inductance limits the upper frequency of operation in a typical PWM converter and reduces the output voltage.

The toroidal core with a homogeneous magnetic material is the closest physical representation to the ideal model of a straight core between magnetic endplates. Since leakage effects are concentrated around the windings, the authors feel that the model would be suitible for any core with a closed magnetic path, such as a pot core.

Appendix

Core radius	10.8 mm
Number of turns	25
Coil inside radius	15 mm
Coil outside radius	25 mm
Coil width	10 mm
Magnetic path length	250 mm
Relative permeability	75
Core effective resistivity	0.01 Ω-m

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William Gerard Hurley (M'77-SM'90) was born in Cork, Ireland in 1952. He received the B.E. degree in electrical engineering from the National University of Ireland (NUI), Cork in 1974, the M.Sc. degree in electrical engineering from the Massachusetts Institute of Technology in 1976 and the Ph.D. degree from the National University of Ireland (NUI), Galway in 1988.

He worked for Honeywell Controls in Canada as a product engineer from 1977 to 1979 and as a development engineer in transmission lines at Ontario Hydro from 1979 to 1983. He lectured in electronic engineering at the University of Limerick, Ireland from 1983 to 1991 and is currently a senior lecturer in Electrical Engineering at University College, Galway, Ireland. He is the director of the Power Electronics Research Center there.

Dr. Hurley is a member of the Administrative Committee of the Power Electronics Society of the IEEE (1993) in New York and a member of Sigma Xi.

David John Wilcox (M'81) was born in Essex, England in 1943. He graduated from the University of Manchester Institute of Science and Technology with the degree of B.Sc. Tech. in 1964 and obtained the degrees of M.Sc. and Ph.D. from the same institution in 1966 and 1969, respectively.

He has held academic appointments at the Faculté Polytechnique de Mons (1969–72), the University of Singapore (1972–76), Monash University (1976–80), and the National University of Ireland (1980–present). He currently holds the Chair of Electronic Engineering at University College, Galway, Ireland.

Professor Wilcox is a Fellow of the Institution of Engineers of Ireland, a Member of the Institution of Electrical Engineers (London) and a Chartered Engineer (U.K.).