Mutual Inductance & Transformers

If a current , i_1 , flows in a coil or circuit then it produces a magnetic field. Some of the magnetic flux may link a second coil of circuit. That flux linkage, Ψ_{21}, will be proportional to the current i_1 providing that any magnetisable material in the neighbourhood is not out of its linear range. We then write

$$\Psi_{21} = M_{21}\, i_1$$

where M_{21}< the constant of proportionality, is the **mutual inductance**.

If the current, i_1 is changing the voltage induced in there is a voltage induced in the second coil,

$$e_2 = -\frac{d\Psi_{21}}{dt} = -M_{21}\frac{di_1}{dt}$$

This is analogous to the induced voltage in the first coil due to the same changing current.

$$e_1 = -L_1\frac{di_1}{dt}$$

where L is the self inductance

If the second coil carries current i_2 then we may write the flux linkage in the first coil as

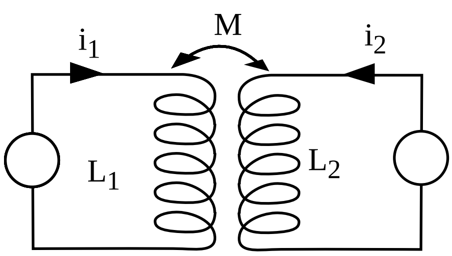
$$\Psi_{12} = M_{12}\, i_2$$

It can be shown from energy considerations (see Appendix) that

$$M_{21} = M_{12} = M$$

The mutual inductance, M, between two coils is

$$\mathbf{M = \frac{Flux\ linkage\ in\ one\ coil}{current\ in\ the\ other\ coil}}$$



The unit of mutual inductance is the same as that of self inductance , namely the henry, H.

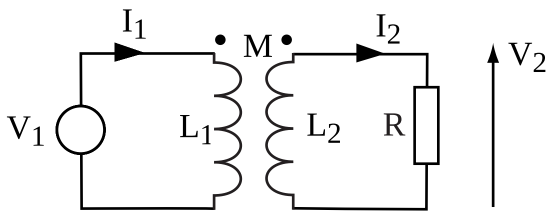
Thus if the currents in the two coils are I_1 and I_2 the induced voltages are

$$e_1 = -L_1\frac{di_1}{dt} - M\frac{di_2}{dt}; \quad e_2 = -M\frac{di_1}{dt} - L_2\frac{di_2}{dt}$$

NOTE: L is always positive: M may have either sign; M may be zero even for coils in close proximity. Think of two partly overlapping circular coils.

Dot notation

Consider a source feeding a load through a mutual inductance.



The dots at the end of the coil symbol signify the 'positive ' end of each coil. If both coils were wound on the same former they would each be wound in the same direction if their helices advanced along the former in the same direction.

In the circuit shown we have for KVL in the frequency-domain on the supply side

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \qquad \ldots 1$$

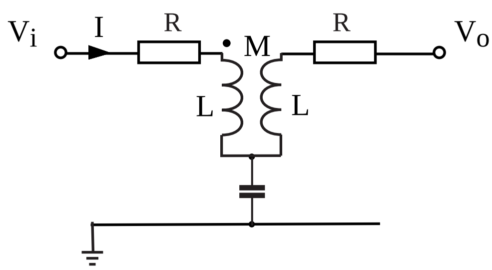
The negative sign for the coefficient of the second term is because I_2 is flowing out of the Dot or positive end.

KVL for the load circuit is

$$0 = - j\omega M I_1 + j\omega L_2 I_2 + I_2 R \qquad \ldots 2$$

Below we show the development of these equations to transformer theory. Meanwhile here is an example in signal theory.

Problem



The filter above is to be a band pass filter. There is no load on the output. Where should the second dot be placed - and why?

Solution

Suppose the second dot is at the top.

KVL for input circuit: $V_i = \left(R + sL + \frac{1}{sC}\right)I$

KVL for output circuit: $V_o = \left(sM + \frac{1}{sC}\right)I$

Combine these: $V_o = \dfrac{sM + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \dfrac{1 + s^2MC}{1 + sRC + s^2LC}\ V_i$

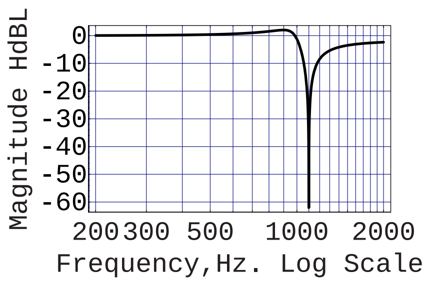
so the transfer function $H(s) = \dfrac{V_o}{V_i} = \dfrac{1 + s^2MC}{1 + sRC + s^2LC}$

The transfer function, $\dfrac{1 + s^2MC}{1 + sRC + s^2LC}$, has zeros at $s = \pm j\sqrt{\dfrac{1}{MC}}$ which lie on the imaginary axis of the s-plane. So the filter is a notch filter; i.e., also a band stop filter. Since M < L the two poles will be at rather lower frequencies where there may be amplification. At zero frequency and infinite frequency $V_o = V_1$: there is no attenuation.

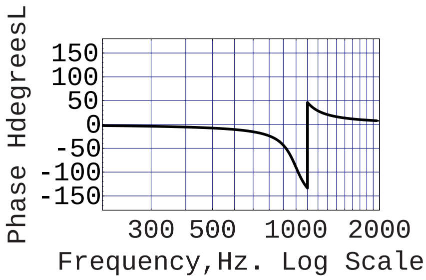
Numerical example. Rewrite the transfer function putting

$$\omega_m = \sqrt{\frac{1}{MC}} \; : \quad \omega_o = \sqrt{\frac{1}{LC}} \quad \frac{r}{L} = \frac{\omega_o}{Q}$$

so $$H(s) = \left(\frac{\omega_o}{\omega_m}\right)^2 \frac{s^2 + \omega_m^2}{s^2 + s\dfrac{\omega_o}{Q} + \omega_o^2}$$



Then set ω_o = 1 kHz, ω_m = 1·1 kHz and Q = 5. The Bode plot becomes that shown opposite.



Note that because we made Q as high as 5 there is a slight amplification just before the notch.

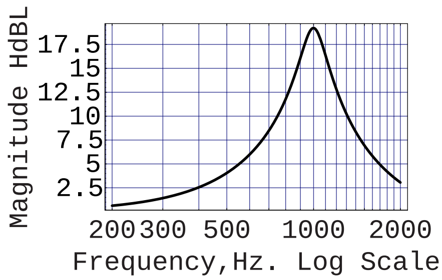
If the second dot were at the bottom KVL for the output circuit would be:

$$V_o = \left(-sM + \frac{1}{sC}\right)$$

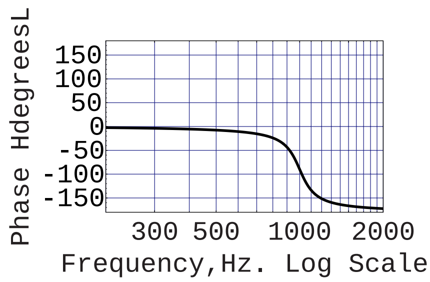
and the transfer function would be

$$H(s) = \frac{1 - s^2MC}{1 + sRC + s^2LC}$$

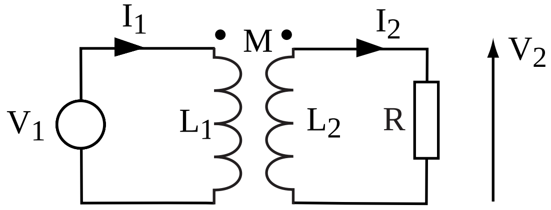
This has zeros at $s = \pm\sqrt{\dfrac{1}{MC}}$



These are both on the real axis and will have only a modifying effect with no pronounced character. There is amplification near the poles but still at the extremes of frequency there is no attenuation/



Transformer



We showed earlier that

KVL in the frequency-domain on the supply side

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \qquad \ldots 1$$

KVL for the load circuit is

$$0 = - j\omega M I_1 + j\omega L_2 I_2 + I_2 R \qquad \ldots 2$$

It is convenient to introduce $V_2 = I_2R$, the voltage across the load and write eq. 2 as

$$0 = - j\omega M I_1 + j\omega L_2 I_2 + V_2$$

so $$V_2 = j\omega M I_1 - j\omega L_2 I_2 \qquad \ldots 2a$$

On no-load when $I_2 = 0$

$$V_2 \equiv V_{2oc} = j\omega M I_1 = V_1 \frac{M}{L_1} \qquad \ldots 3$$

where V_{2oc} is the **open circuit secondary voltage.** The load terminals are called the secondary side or secondary terminals: the supply side are the primary terminals.

Also on no-load when $I_2 = 0$ we get from eq. 1

$$I_1 \equiv I_M = \frac{V_1}{j\omega L_1} \qquad \ldots 4$$

where I_M is called the **magnetising current.**

From eq. 1 we have in general

$$I_1 = \frac{V_1}{j\omega L_1} + \frac{M}{L_1} I_2 = I_M + \frac{M}{L_1} I_2 \qquad \ldots 5$$

Combine eq. 5 with eq. 2a so that on load

$$V_2 = j\omega M\left(\frac{V_1}{j\omega L_1} + \frac{M}{L_1} I_2\right) - j\omega L_2 I_2 = V_1 \frac{M}{L_1} - j\omega L_2\left(1 - \frac{M^2}{L_1 L_2}\right) I_2$$

$$= V_{2oc} - j\omega X_2 I_2 \qquad \ldots 6$$

with $$X_2 = \omega L_2\left(1 - \frac{M^2}{L_1 L_2}\right) \qquad \ldots 7$$

X_2 is **the leakage reactance referred to the load or secondary side**.

Another important feature of the mutual inductance between two circuits is, as shown in the Appendix, that

$$\mathbf{M^2 \leq L_1 L_2}$$

We define a **coupling coefficient, k,** such that

$$M = k \sqrt{L_1 L_2}$$

For two coils wound close to each other the coupling coefficient, k, will be close to unity. The factor $\left(1 - \frac{M^2}{L_1 L_2}\right) = (1 - k^2)$ in eq. 7 will be small. Anyway we may write eq. 7

$$X_2 = \omega L_2(1 - k^2) \qquad \text{...7a}$$

A mutual inductor is a transformer though the latter term, transformer, is usually reserved for the arrangement where there is a magnetic circuit completely formed of iron or other ferromagnetic material and the two coils are wound onto this common magnetic circuit usually called the core.

Now from electromagnetic theory we have for coils wound on magnetic circuits

$$L_1 \approx \frac{N_1^2}{R_M} \qquad L_2 \approx \frac{N_2^2}{R_M} \qquad \text{...8}$$

where R_M is the **reluctance** of the magnetic circuit. For ferromagnetic circuits

$$R_M = \frac{l}{A\, \mu_r \mu_o} \qquad \text{...9}$$

where l is the effective (usually average) length of the magnetic circuit, A is its cross-sectional area, μ_o is the magnetic constant ($4\pi\ 10^{-7}$ H m^{-1}) and μ_r is the relative permeability of the ferromagnetic material from which the core is made.

Now $\quad M^2 = k^2 L_1 L_2 = k^2 \frac{N_1^2 N_2^2}{R_M^2} \quad$ so $\quad M \approx k \frac{N_1 N_2}{R_M} \qquad$ ...10

We found the open circuit secondary voltage: $\quad V_{2oc} = V_1 \frac{M}{L_1} \qquad$ ...3

Thus taking k as close to unity from eqq. 8 and 10, $\quad \frac{M}{L_1} \approx \frac{N_2}{N_1}$. Thus from eq. 3

$$V_{2oc} = V_1 \frac{M}{L_1} \approx V_1 \frac{N_2}{N_1} \qquad \text{...11}$$

which is the simple relationship normally taken for a transformer. Although the theory is approximate the relationship is quite close in practice.

$$\frac{V_{2oc}}{V_1} = \frac{N_2}{N_1} \qquad \text{...10a}$$

Similarly from eq. 5 $I_1 = I_M + \frac{M}{L_1} I_2$ …5

$$= I_M + \frac{N_2}{N_1} I_2 \quad \text{…12}$$

$$= I_M + I_{1load} \quad \text{…12a}$$

with $\frac{I_{1load}}{I_2} = \frac{N_2}{N_1}$ …12b

In **summary** these following relationships for a transformer are widely used

$$\frac{V_{2oc}}{V_1} = \frac{N_2}{N_1} \quad \text{…10a} \qquad \frac{I_{1load}}{I_2} = \frac{N_2}{N_1} \quad \text{…12b} \qquad V_2 = V_{2oc} - j\omega X_2 I_2 \quad \text{…6}$$

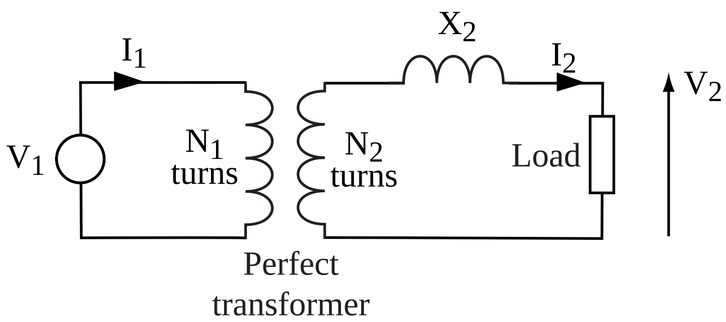
$$I_M = \frac{V_1}{j\omega L_1} \quad \text{…4} \qquad I_1 = I_M + \frac{N_2}{N_1} I_2 \quad \text{…12a}$$

The leakage reactance referred to the secondary;

$$X_2 = \omega L_2(1 - k) \quad \text{…7a}$$

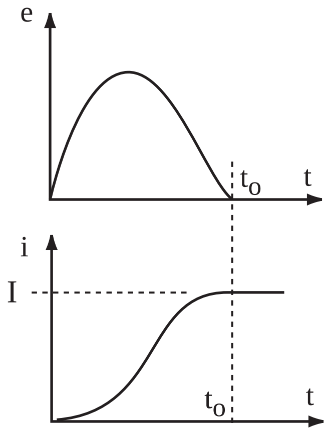
The I_M is the easy to compute at least to a rough order using the formulae for reactance given above; eqq. 7 & 8. The computation of X_2 is fairly complicated since k must be determined quite accurately; finding X_2 does not figure in 2nd year exam papers depending as it does on detailed geometry of the core and windings.

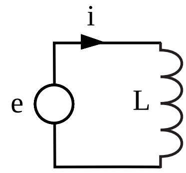
Equations 10a, 11b and 6 allow the construction of a simple equivalent circuit:–



Appendix : Energy Theorems

Energy stored in a coil





Look at the charging process. When the generator voltage is positive and the current through the inductor increases

$$e = L \frac{di}{dt}$$

When the voltage falls to zero the inductor is charged and the current remains constant. Note that there is no resistance in this circuit.

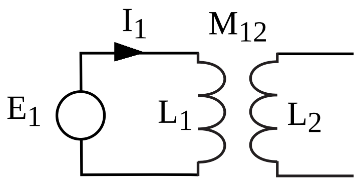
The power, p = ei, and therefore the energy input during charging is

$$W = \int_0^{t_o} p\, dt = \int_0^{t_o} e\, i\, dt = \int_0^{t_o} i\, L \frac{di}{dt}\, dt = \int_0^{I} i\, L\, di = \frac{1}{2} L I^2$$

This is the stored energy $W = \frac{1}{2} L i^2$ …13

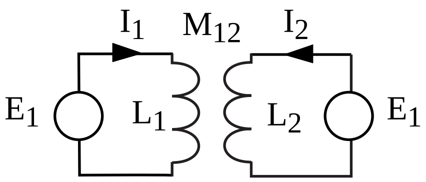
To show that $M_{12}=M_{21}$ by an energy method

Consider a pair of coils of self inductance L_1 and L_2.



Stage 1 Charge up the first coil with a generator to a current I_1. The stored energy is

$$W_1 = \frac{1}{2} L_1 I_1^2 \qquad \ldots 14$$



Stage 2. Keeping the current in coil 1 constant charge up the second coil to a current I_2.

The only voltages induced in coil 2 in this process are those due to the fields caused by current I_2.

Thus the energy supplied by the generator E_2 is

$$W_2 = \frac{1}{2} L_2 I_2^2 \qquad \ldots 15$$

However there will be voltages induced in coil 1 by the changing current in coil 2 of

$$V_{12} = -M_{12}\frac{di_2}{dt}$$

where i_2 is a value of current during the charging process.

To keep I_1 at a constant value the generator E_1 must counteract this voltage and in so doing will exchange energy at a rate

$$\frac{dW_{12}}{dt} = -V_{12} I_1$$

The total energy exchanged with generator E_1 during the raising of current in coil 2 is

$$W_{12} = \int dW_{12} = \int \frac{dW_{12}}{dt} dt = \int -V_{12} I_1 \, dt$$

$$= \int I_1 M_{12} \frac{di_2}{dt} dt = \int_0^{I_2} I_1 M_{12} \, di_2$$

$$= M_{12} I_1 I_2 \qquad \ldots 16$$

The energy exchange may be positive or negative depending on whether M_{12} is positive or negative , that is on whether the field due to I_2 strengthens (when E_1 supplies energy) or weakens the field due to I_1. Take the positive value now.

The total energy supplied by the generators is the sum of eqq. 14, 15 & 16

$$W_1 + W_2 + W_{12} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

This is the total stored energy in the magnetic field. It can be extracted by reversing the process.

If we had begun by charging coil 2 first and then coil 1 the total stored energy would have been

$$W = W_1 + W_2 + W_{21} = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$$

Since the two final conditions are indistinguishable the final stored energy must be the same however the charging was done (else we could charge by one process and discharge by the other and thereby defeat the first law of thermodynamics, a trick which is not possible). The two expressions for stored energy must be the same. Thus

$$W_{12} = W_{21}$$

$$\text{and} \quad M_{12} = M_{21} = M \qquad \ldots 17$$

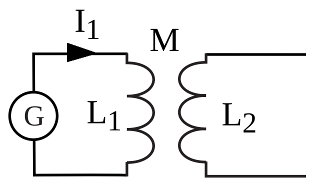
Mutual inductance is reciprocal. There is only one mutual inductance for a pair of coils.

Thus the **total stored energy** in the total magnetic field due to the currents in both coils is

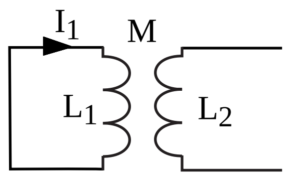
$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + I_1 I_2 M. \qquad \ldots 18$$

NOTE The mutual inductance, M, may be positive or negative. If is negative then current in one coil tends to cancel the magnetic flux due to current in the other coil.

To show that $M^2 \le L_1 L_2$

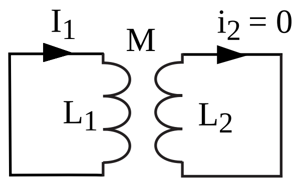


Step 1. Charge up the magnetic field in Inductor 1 so that a current I_1 flows. Inductor 2 is on open circuit

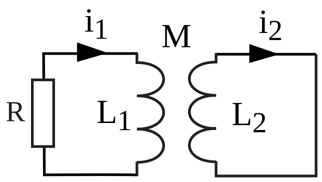


Step 2 Short circuit Inductor 1. The current will continue to flow there being no resistance. The stored energy is

$$\frac{1}{2} L_1 I_1^2$$



Step 3 Short circuit Inductor 2. There is no electromagnetic change.



Step 4 Open the circuit of Inductor 1. This can be done by inserting a variable resistance and increasing its value to infinity until no current flows in Inductor 1. This can be done very quickly though the rate does not matter as we shall see.

The differential equation governing the current in Inductor 2 during Step 4 is

$$L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = 0$$

Integrate this with regard to time for Step 4:-

$$\int_0^t L_2 \frac{di_2}{dt} dt + \int_0^t M \frac{di_1}{dt} dt = 0$$

or, changing variables,

$$\int_0^{I_2} L_2 \, di_2 + \int_{I_1}^0 M \, di_1 = 0$$

where I_2 is the final value of i_2

Thus

$$L_2 I_2 - M I_1 = 0$$

and the current in coil 2 after Step 4 is

$$I_2 = I_1 \frac{M}{L_2}$$

The final stored energy is all in coil 2 and is

$$\frac{1}{2} L_2 I_2^2 = \frac{1}{2} L_2 \left(\frac{I_1 M}{L_2} \right)^2$$

$$= \frac{1}{2} \frac{M^2}{L_2} I_1^2$$

This must be less than or equal to the energy stored in Inductor 1 just after it had been charged. The first law of thermodynamics holds: all the excess energy is dissipated in the resistor or radiated.

Thus

$$\frac{1}{2} \frac{M^2}{L_2} I_1^2 \le \frac{1}{2} L_1 I_1^2$$

or $$M^2 \le L_1 L_2 .$$

We often write

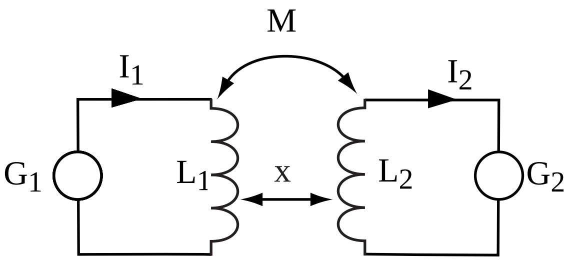
$$M^2 = k^2 L_1 L_2 \qquad \ldots 19$$

where k is called the **coupling coefficient. $k \le 1$.**

NOTE some authors define the coupling coefficient by $M^2 = k L_1 L_2$

Force between two current carrying coils

Consider again a pair of coils of self inductances L_1 and L_2 and of mutual inductance M and carrying currents I_1 and I_2. The supplies to the two coils keep the currents constant.



We are now going to calculate the force between the coils in the x-direction.

Suppose that the coils move apart against the force F_x by a distance δx in time δt.

Stage 1. The work done against the force, F_x , is $W_x = F_x\,\delta x$.

Stage 2. There is a change in flux in circuit 1, $\delta\Psi = I_2\,\delta M = I_2 \frac{dM}{dx}\,\delta x.$

The voltage induced in circuit 1 is $V_1 = -\frac{\delta\Psi}{\delta t} = -I_2 \frac{dM}{dx}\frac{\delta x}{\delta t}$.

The power provided by the supply G1, $P_1 = -I_1 V_1 = I_1 I_2 \frac{dM}{dx}\frac{\delta x}{\delta t}$.

Thus the energy supplied, $W_1 = P_1\,\delta t = I_1 I_2 \frac{dM}{dx}\,\delta x = I_1 I_2\,\delta M.$

Stage 3. Similarly the energy supplied by G2 is the same:– $W_2 = I_1 I_2\,\delta M.$

Stage 4. The increase in stored energy of the coils is:–

$$W_s = \frac{d}{dx}\left\{\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + I_1 I_2 M\right\}\delta x = I_1 I_2\,\delta M.$$

Stage 5. The energy supplied by the electric power supplies, G1 and G2, to sustain the current goes partly into increasing the stored energy in the magnetic field and partly into moving the constraining force:

$$W_1 + W_2 = W_s + W_x$$

$$2\,I_1 I_2\,\delta M = I_1 I_2\,\delta M + F_x\,\delta x$$

Whence $$F_x = I_1 I_2 \frac{dM}{dx} \qquad \ldots 20$$

Now the flux linking circuit 2 due to the current in **1** is $\Psi_{21} = I_1 M$;

thus $$F_x = I_2 \frac{d\Psi_{21}}{dx} = I_1 \frac{d\Psi_{12}}{dx} . \qquad \ldots 21$$

This force is independent of the source of the flux; it could be from a permanent magnet.