CHAPTER 2

ELECTROMAGNETIC THEORY

2.1 MAXWELL'S EQUATIONS

Electric and magnetic fields that vary with time are governed by physical laws described by a set of equations known collectively as Maxwell's equations. For the most part these equations were arrived at from experiments carried out by several investigators. It is not our purpose here to justify the basis for these equations, but rather to gain some understanding of their physical significance and to learn how to obtain solutions of these equations in practical situations of interest in the microwave engineering field. The electric field $\boldsymbol{\mathscr{E}}$ and magnetic field $\boldsymbol{\mathscr{B}}$ are vector fields and in general have amplitudes and directions that vary with the three spatial coordinates x, y, z and the time coordinate t.† In mks units, which are used throughout, the electric field is measured in volts per meter and the magnetic field in webers per square meter. Since these fields are vector fields, the equations governing their behavior are most conveniently written in vector form.‡

The electric field $\boldsymbol{\mathscr{E}}$ and magnetic field $\boldsymbol{\mathscr{B}}$ are regarded as fundamental in that they give the force on a charge q moving with velocity $\mathbf{v}$; that is,

$$\mathbf{F} = q(\boldsymbol{\mathscr{E}} + \mathbf{v} \times \boldsymbol{\mathscr{B}}) \tag{2.1}$$

†Boldface script type is used to represent vector fields having arbitrary time dependence. Boldface roman type is used later for the phasor representation of fields having sinusoidal time dependence.

‡It is assumed that the reader is familiar with vector analysis. However, for convenient reference, a number of vector formulas and relations are summarized in App. I.

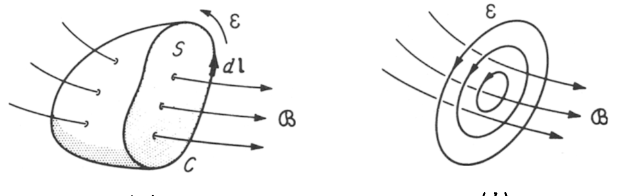

FIGURE 2.1
Illustration of Faraday's law.

where $\mathbf{F}$ is the force in newtons, q is the charge measured in coulombs, and $\mathbf{v}$ is the velocity in meters per second. This force law is called the Lorentz force equation. In addition to the $\mathscr{E}$ and $\mathscr{B}$ fields, it is convenient to introduce two auxiliary field vectors, namely, the electric displacement $\mathscr{D}$ and the magnetic intensity $\mathscr{H}$. These are related to $\mathscr{E}$ and $\mathscr{B}$ through the electric and magnetic polarization of material media, a topic covered in the next section. In this section we consider fields in vacuum, or *free space*, only. In this case the following simple relationships hold:

$$\mathscr{H} = \frac{1}{\mu_0}\mathscr{B} \tag{2.2a}$$

$$\mathscr{D} = \epsilon_0 \mathscr{E} \tag{2.2b}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m and is called the permeability of vacuum, and $\epsilon_0 = 10^{-9}/36\pi = 8.854 \times 10^{-12}$ F/m and is known as the permittivity of vacuum.

One of the basic laws of electromagnetic phenomena is Faraday's law, which states that a time-varying magnetic field generates an electric field. With reference to Fig. 2.1, let C denote an arbitrary closed curve that forms the boundary of a nonmoving surface S. The time rate of change of total magnetic flux through the surface S is $\partial(\int_S \mathscr{B} \cdot d\mathbf{S})/\partial t$. According to Faraday's law, this time rate of change of total magnetic flux is equal to the negative value of the total voltage measured around C. The later quantity is given by $-\oint_C \mathscr{E} \cdot d\mathbf{l}$. Hence the mathematical statement of Faraday's law is

$$\oint_C \mathscr{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathscr{B} \cdot d\mathbf{S} \tag{2.3}$$

The line integral of $\mathscr{E}$ around C is a measure of the circulation, or "curling up," of the electric field in space. The time-varying magnetic field may be properly regarded as a vortex source that produces an electric field having nonzero curl, or circulation. Although (2.3) is in a form that is readily interpreted physically, it is not in a form suitable for the analysis of a physical problem. What is required is a differential equation that is equivalent to (2.3). This equation may be obtained by using Stokes' theorem from vector analysis, which states that the line integral of a vector around a closed contour C is equal to the integral of the normal component of the curl of this vector over any surface having C as its boundary. The curl of a vector is written $\nabla \times \mathscr{E}$ (App. I), and hence (2.1) becomes

$$\oint_C \mathscr{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathscr{E} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_S \mathscr{B} \cdot d\mathbf{S}$$

Since S is completely arbitrary, the latter two integrals are equal only if

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} \tag{2.4}$$

which is the desired differential equation describing Faraday's law. The curl is a measure of the circulation of a vector field at a point.

Helmholtz's theorem from vector analysis states that a vector field is completely defined only when the curl, or circulation, of the field, and also its divergence, are given at every point in space. Now the divergence (or convergence) of field lines arises only if a proper source (or sink) is available. The electric field, in addition to having a curl produced by the vortex source $-\partial\mathscr{B}/\partial t$, has a divergence produced by electric charge. Gauss' law states that the total flux of $\mathscr{D} = \epsilon_0\mathscr{E}$ from a volume V is equal to the net charge contained within V. If ρ represents the charge density in coulombs per cubic meter, Gauss' law may be written as

$$\oint_S \epsilon_0\mathscr{E} \cdot d\mathbf{S} = \int_V \rho\, dV \tag{2.5}$$

This equation may be converted to a differential equation by using the divergence theorem to give

$$\oint_S \epsilon_0\mathscr{E} \cdot d\mathbf{S} = \int_V \nabla \cdot \epsilon_0\mathscr{E}\, dV = \int_V \rho\, dV$$

Since V is arbitrary, it follows that

$$\nabla \cdot \epsilon_0\mathscr{E} = \nabla \cdot \mathscr{D} = \rho \tag{2.6}$$

where $\nabla \cdot \mathscr{D}$ is the divergence of $\mathscr{D}$, that is, a measure of the total outward flux of $\mathscr{D}$ from a volume element, divided by the volume of the element, as this volume shrinks to zero. Since both the curl and divergence of the electric field are now specified, this field is completely determined in terms of the two sources, $\partial\mathscr{B}/\partial t$ and ρ.

To complete the formulation of electromagnetic phenomena, we must now relate the curl and divergence of the magnetic field to their sources. The vortex source that creates the circulation, or curl, of the magnetic field $\mathscr{H}$ is the current. By current is meant the total current density, the conduction current density $\mathscr{J}$ measured in amperes per square meter, the displacement current density $\partial\mathscr{D}/\partial t$, and the convection current $\rho\mathbf{v}$ consisting of charge in motion if present. Convection current is not included in this chapter. However, in the chapter dealing with microwave tubes, convection current plays a central role and is discussed in detail there. The displacement current density $\partial\mathscr{D}/\partial t$ was first introduced by Maxwell, and leads to the possibility of wave motion, as will be seen. Mathematically, the circulation of $\mathscr{H}$ around a closed contour C bounding a surface S as in Fig. 2.1 is given by

$$\oint_C \mathscr{H} \cdot d\mathbf{l} = \int_S \frac{\partial \mathscr{D}}{\partial t} \cdot d\mathbf{S} + \int_S \mathscr{J} \cdot d\mathbf{S} \tag{2.7}$$

Application of Stokes' law to the left-hand side yields

$$\int_S \nabla \times \mathscr{H} \cdot d\mathbf{S} = \int_S \frac{\partial \mathscr{D}}{\partial t} \cdot d\mathbf{S} + \int_S \mathscr{J} \cdot d\mathbf{S}$$

from which it may be concluded that

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J} \tag{2.8}$$

Since magnetic charge, as the dual of electric charge, does not exist in nature, it may be concluded that the divergence of $\mathscr{B}$ is always zero; i.e., the flux lines of $\mathscr{B}$ are always closed since there are no charges for them to terminate on. Thus the net flux of $\mathscr{B}$ through any closed surface S is always zero; i.e., just as much flux enters through the surface as leaves it. Corresponding to (2.5) and (2.6), we thus have

$$\oint_S \mathscr{B} \cdot d\mathbf{S} = 0 \tag{2.9}$$

$$\nabla \cdot \mathscr{B} = 0 \tag{2.10}$$

Conduction current, of density $\mathscr{J}$, is the net flow of electric charge. Since charge is conserved, the total rate of flow of charge out of a volume V is equal to the time rate of decrease of total charge within V, as expressed by the equation

$$\oint_S \mathscr{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho \, dV \tag{2.11}$$

This is the continuity equation, and it may be converted to a differential equation by using the divergence theorem in the same manner as was done to derive (2.6) from (2.5). It is readily found that

$$\nabla \cdot \mathscr{J} + \frac{\partial \rho}{\partial t} = 0 \tag{2.12}$$

This equation may also be derived from (2.8) and (2.6). Since the divergence of the curl of any vector is identically zero, the divergence of (2.8) yields

$$0 = \frac{\partial \nabla \cdot \mathscr{D}}{\partial t} + \nabla \cdot \mathscr{J}$$

Using (2.6) converts this immediately into the continuity equation (2.12). If the displacement current density $\partial \mathscr{D}/\partial t$ had not been included as part of the total current density on the right-hand side of (2.8), that equation would have led to the conclusion that $\nabla \cdot \mathscr{J} = 0$, a result inconsistent with the continuity equation unless the charge density was independent of time.

In summary, the four equations, known as Maxwell's equations, that describe electromagnetic phenomena in vacuum are

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} \tag{2.13a}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J} \tag{2.13b}$$

$$\nabla \cdot \mathscr{D} = \rho \tag{2.13c}$$

$$\nabla \cdot \mathscr{B} = 0 \tag{2.13d}$$

where in (2.13b) the convection current $\rho\mathbf{v}$ has not been included. The continuity equation may be derived from (2.13b) and (2.13c), and hence contains no additional information. Although $-\partial\mathscr{B}/\partial t$ may be regarded as a source for $\mathscr{E}$, and $\partial\mathscr{D}/\partial t$ as a source of $\mathscr{H}$, the ultimate sources of an electromagnetic field are the current $\mathscr{J}$ and charge ρ. For time-varying fields, that charge density ρ which varies with time is not independent of $\mathscr{J}$ since it is related to the latter by the continuity equation. As a consequence, it is possible to derive the time-varying electromagnetic field from a knowledge of the current density $\mathscr{J}$ alone.

It is not difficult to show in a qualitative way that (2.13a) and (2.13b) lead to wave propagation, i.e., to the propagation of an electromagnetic disturbance through space. Consider a loop of wire in which a current varying with time flows as in Fig. 2.2. The conduction current causes a circulation, or curling, of the magnetic field around the current loop as in Fig. 2.2a (for clarity only a few flux lines are shown). The changing magnetic field in turn creates a circulating, or curling, electric field, with field lines that encircle the magnetic field lines as in Fig. 2.2b. This changing electric field creates further curling magnetic field lines as in Fig. 2.2c, and so forth. The net result is the continual growth and spreading of the electromagnetic field into all space surrounding the current loop. The

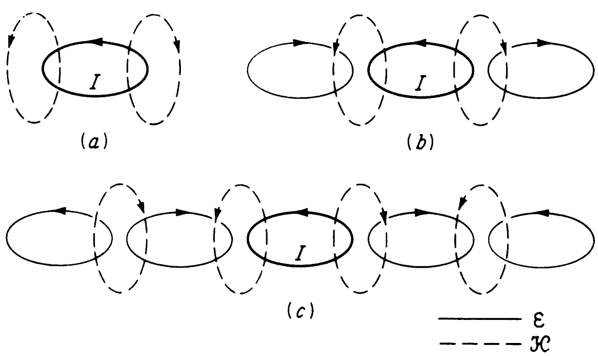

FIGURE 2.2
The growth or generation of an electromagnetic wave from a current loop.

disturbance moves outward with the velocity of light. A little thought will show that the same characteristic mutual effect between two quantities must always exist for wave motion. That is, quantity A must be generated by quantity B, and vice versa. For example, in an acoustical wave the excess pressure creates a motion of the adjacent air mass. The motion of the air mass by virtue of its inertia in turn creates a condensation, or excess pressure, farther along. The repetition of this process generates the acoustical wave.

For the most part, as at lower frequencies, it is sufficient to consider only the steady-state solution for the electromagnetic field as produced by currents having sinusoidal time dependence. The time derivative may then be eliminated by denoting the time dependence of all quantities as $e^{j\omega t}$ and representing all field vectors as complex-phasor space vectors independent of time. Boldface roman type is used to represent these complex-phasor space vectors. For example, the mathematical representation for the electric field $\mathscr{E}(x, y, z, t)$ will be $\mathbf{E}(x, y, z)e^{j\omega t}$. Each component of $\mathbf{E}$ is in general complex, with a real and imaginary part; thus

$$\mathbf{E} = \mathbf{a}_x(E_{xr} + jE_{xi}) + \mathbf{a}_y(E_{yr} + jE_{yi}) + \mathbf{a}_z(E_{zr} + jE_{zi}) \qquad (2.14)$$

where the subscript r refers to the real part and the subscript i refers to the imaginary part. Each component is allowed to be complex in order to provide for an arbitrary time phase for each component. This may be seen by recalling the usual method of obtaining $\mathscr{E}$ from its phasor representation. That is, by definition,

$$\mathscr{E}(x, y, z, t) = \mathrm{Re}\left[\mathbf{E}(x, y, z)e^{j\omega t}\right] \qquad (2.15)$$

Thus

$$\begin{aligned} E_x &= \mathrm{Re}\left[(E_{xr} + jE_{xi})e^{j\omega t}\right] \\ &= \mathrm{Re}\left(\sqrt{E_{xr}^2 + E_{xi}^2}\, e^{j\omega t + j\phi}\right) \\ &= \sqrt{E_{xr}^2 + E_{xi}^2}\cos(\omega t + \phi) \end{aligned}$$

where $\phi = \tan^{-1}(E_{xi}/E_{xr})$. Unless E_x had both an imaginary part jE_{xi} and a real part E_{xr}, the arbitrary phase angle ϕ would not be present. As a general rule, the time factor $e^{j\omega t}$ will not be written down when the phasor representation is used. However, it is important to remember both the fact that such a time dependence is implied and also the rule (2.15) for obtaining the physical field vector from its phasor representation. The real and imaginary parts of the space components of a vector should not be confused with the space components; for example, E_{xr} and E_{xi} are not two space components of E_x since the component $\mathbf{a}_x E_x$ is always directed along the x axis in space, with the real and imaginary parts simply accounting for an arbitrary time phase or origin.

A further point of interest in connection with the phasor representation is the method used for obtaining the time-average value of a field quantity.

For example, if

$$\mathscr{E} = \mathbf{a}_x E_1 \cos(\omega t + \phi_1) + \mathbf{a}_y E_2 \cos(\omega t + \phi_2) + \mathbf{a}_z E_3 \cos(\omega t + \phi_3)$$

the time-average value of $|\mathscr{E}|^2$ is

$$\begin{aligned} |\mathscr{E}|^2_{\text{av}} &= \frac{1}{T}\int_0^T \mathscr{E}\cdot\mathscr{E}\,dt \\ &= \frac{1}{T}\int_0^T \left[E_1^2\cos^2(\omega t + \phi_1) + E_2^2\cos^2(\omega t + \phi_2)\right. \\ &\qquad\qquad \left. + E_3^2\cos(\omega t + \phi_3)\right] dt \\ &= \frac{1}{2}\left(E_1^2 + E_2^2 + E_3^2\right) \end{aligned} \tag{2.16}$$

where T is the period, equal to $2\pi/\omega$. The same result is obtained by simply taking one-half of the scalar, or dot, product of $\mathbf{E}$ with the complex conjugate $\mathbf{E}^*$; thus

$$|\mathscr{E}|^2_{\text{av}} = \tfrac{1}{2}\mathbf{E}\cdot\mathbf{E}^* = \tfrac{1}{2}\left[\left(E_{xr}^2 + E_{xi}^2\right) + \left(E_{yr}^2 + E_{yi}^2\right) + \left(E_{zr}^2 + E_{zi}^2\right)\right] \tag{2.17}$$

since $E_x E_x^* = (E_{xr} + jE_{xi})(E_{xr} - jE_{xi}) = E_{xr}^2 + E_{xi}^2$, etc. This is equal to (2.16), since $E_1^2 = E_{xr}^2 + E_{xi}^2$, etc.

By using the phasor representation, the time derivative $\partial/\partial t$ may be replaced by the factor $j\omega$ since $\partial e^{j\omega t}/\partial t = j\omega e^{j\omega t}$. Hence Maxwell's equations, with steady-state sinusoidal time dependence, become

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \tag{2.18a}$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} \tag{2.18b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2.18c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.18d}$$

2.2 CONSTITUTIVE RELATIONS

In material media the auxiliary field vectors $\mathscr{H}$ and $\mathscr{D}$ are defined in terms of the polarization of the material and the fundamental field quantities $\mathscr{B}$ and $\mathscr{E}$. The relationships of $\mathscr{H}$ to $\mathscr{B}$ and of $\mathscr{D}$ to $\mathscr{E}$ are known as constitutive relations, and must be known before solutions for Maxwell's equations can be found.

Consider first the electric case. If an electric field $\mathscr{E}$ is applied to a material body, this force results in a distortion of the atoms or molecules in such a manner as to create effective electric dipoles with a dipole moment $\mathscr{P}$ per unit volume. The total displacement current is the sum of the vacuum displacement current $\partial\epsilon_0\mathscr{E}/\partial t$ and the polarization current $\partial\mathscr{P}/\partial t$. To avoid accounting for the polarization current $\partial\mathscr{P}/\partial t$ explicitly, the

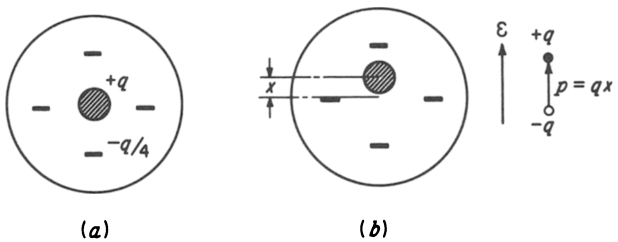

FIGURE 2.3
Model for determining the polarization of an atom.

displacement vector $\mathscr{D}$ is defined as

$$\mathscr{D} = \epsilon_0 \mathscr{E} + \mathscr{P} \tag{2.19}$$

whence the total displacement current density can be written as $\partial \mathscr{D} / \partial t$.

For a great many materials the polarization $\mathscr{P}$ is in the direction of the electric field $\mathscr{E}$, although rarely will $\mathscr{P}$ have the same time phase as $\mathscr{E}$. A simple classical model will serve to illustrate this point. Figure 2.3*a* shows a model of an atom consisting of a nucleus with charge q surrounded by a spherically symmetrical electron cloud of total charge $-q$. The application of a field $\mathscr{E}$ displaces the electron cloud an effective distance x as in Fig. 2.3*b*. This displacement is resisted by a restoring force kx proportional to the displacement (Prob. 2.1). In addition, dissipation, or damping, effects are present and result in an additional force, which we shall assume to be proportional to the velocity. If m is the effective mass of the electron cloud, the dynamical equation of motion is obtained by equating the sum of the inertial force $m\, d^2x/dt^2$, viscous force $m\nu\, dx/dt$, and restoring force kx to the applied force $-q\mathscr{E}$; thus

$$m \frac{d^2x}{dt^2} + m\nu \frac{dx}{dt} + kx = -q\mathscr{E} \tag{2.20}$$

When $\mathscr{E} = E_x \cos \omega t$, the solution for x is of the form $x = -A \cos(\omega t + \phi)$.

If $E_x \cos \omega t$ is represented by the phasor E_x, and x by the phasor X, the solution for X is readily found to be

$$X = \frac{-qE_x}{-\omega^2 m + j\omega\nu m + k}$$

and hence

$$x = \mathrm{Re}(Xe^{j\omega t}) = A \cos(\omega t + \phi)$$

where

$$A = \frac{(q/m)E_x}{\left[\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu^2\right]^{1/2}}$$

$$\phi = \tan^{-1} \frac{\omega\nu}{\omega^2 - \omega_0^2}$$

and we have replaced k/m by ω_0^2.

The dipole moment is p_x, where

$$p_x = -qx = \frac{q^2 E_x}{m\left[\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu^2\right]^{1/2}} \cos(\omega t + \phi) \tag{2.21}$$

For N such atoms per unit volume the polarization per unit volume is $\mathscr{P}_x = Np_x$ and the displacement $\mathscr{D}_x$ is given by

$$\mathscr{D}_x = \epsilon_0 E_x \cos \omega t + \frac{Nq^2 E_x}{m\left[\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu^2\right]^{1/2}} \cos(\omega t + \phi)$$

This equation may also be put into the following form:

$$\mathscr{D}_x = E_x \left\{ \frac{\left[\epsilon_0\left(\omega_0^2 - \omega^2\right) + Nq^2/m\right]^2 + \left(\omega\nu\epsilon_0\right)^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\omega\nu\right)^2} \right\}^{1/2} \cos(\omega t - \theta) \tag{2.22}$$

where $\theta = \tan^{-1} \dfrac{\omega\nu}{\omega_0^2 - \omega^2} - \tan^{-1} \dfrac{\omega\nu}{\omega_0^2 - \omega^2 + Nq^2/\epsilon_0 m}$

Two points are of interest in connection with (2.22). One is the linear relationship between $\mathscr{P}$ and $\mathscr{E}$, and hence between $\mathscr{D}$ and $\mathscr{E}$. The second is the phase lag in $\mathscr{D}$ relative to $\mathscr{E}$ whenever damping forces are present.

The phase difference between $\mathscr{P}$, $\mathscr{E}$, and $\mathscr{D}$ makes it awkward to handle the relations between these quantities unless phasor representation is used. In phasor representation (2.21) and (2.22) become

$$P_x = \frac{q^2 E_x}{\left(\omega_0^2 - \omega^2 + j\omega\nu\right)m} \tag{2.23}$$

$$D_x = \frac{\epsilon_0\left(\omega_0^2 - \omega^2 + j\omega\nu\right) + Nq^2/m}{\omega_0^2 - \omega^2 + j\omega\nu} E_x \tag{2.24}$$

In general, for linear media, we may write

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{2.25}$$

where χ_e is a complex constant of proportionality called the electric susceptibility. The equation for $\mathbf{D}$ becomes

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon \mathbf{E} = \epsilon_r \epsilon_0 \mathbf{E} = (\epsilon' - j\epsilon'')\mathbf{E} \end{aligned} \tag{2.26}$$

where $\epsilon = \epsilon_0(1 + \chi_e)$ is called the permittivity, and $\epsilon_r = \epsilon/\epsilon_0$, the dielectric constant of the medium. Note that ϵ is complex whenever damping effects are present and that the imaginary part is always negative. A positive imaginary part would imply energy creation instead of energy loss. [The reader may verify from (2.22) that θ is always positive.]

Loss in a dielectric material may also occur because of a finite conductivity σ. The two mechanisms are indistinguishable as far as external effects related to power dissipation are concerned. The curl equation for **H** may be written as

$$\nabla \times \mathbf{H} = j\omega(\epsilon' - j\epsilon'')\mathbf{E} + \sigma\mathbf{E}$$

where $\mathbf{J} = \sigma\mathbf{E}$ is the conduction current density in the material. We may also write

$$\nabla \times \mathbf{H} = j\omega\left[\epsilon' - j\left(\epsilon'' + \frac{\sigma}{\omega}\right)\right]\mathbf{E} = j\omega\epsilon'\mathbf{E} + (\omega\epsilon'' + \sigma)\mathbf{E} \tag{2.27}$$

where by $\epsilon'' + \sigma/\omega$ may be considered as the effective imaginary part of the permittivity, or $\omega\epsilon'' + \sigma$ as the total effective conductivity.

The loss tangent of a dielectric medium is defined by

$$\tan \delta_l = \frac{\omega\epsilon'' + \sigma}{\omega\epsilon'} \tag{2.28}$$

Any measurement of $\tan \delta_l$ always includes the effects of finite conductivity σ. At microwave frequencies, however, $\omega\epsilon''$ is usually much larger than σ because of the large value of ω.

Materials for which **P** is linearly related to **E** and in the same direction as **E** are called linear isotropic materials. Nonlinear effects generally occur only for very large applied fields, and as a consequence are rarely encountered in microwave work. However, nonisotropic material is of some importance. If the crystal structure lacks spherical symmetry such as that in a cubic crystal, it may be anticipated that the polarization per unit volume will depend on the direction of the applied field. In Fig. 2.4 a two-dimensional sketch of a crystal lacking cubic symmetry is given. The polarization produced when the field is applied along the x axis may be greater than that when the field is applied along the y or z axis because of the greater ease of polarization along the x axis. In this case we must write

$$D_x = \epsilon_{xx}E_x \qquad D_y = \epsilon_{yy}E_y \qquad D_z = \epsilon_{zz}E_z \tag{2.29}$$

where ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} are, in general, all different. The dielectric constants $\epsilon_{rx} = \epsilon_{xx}/\epsilon_0$, $\epsilon_{ry} = \epsilon_{yy}/\epsilon_0$, $\epsilon_{rz} = \epsilon_{zz}/\epsilon_0$ are known as the principal dielectric constants, and the material is said to be anisotropic. If the coordinate system used had a different orientation with respect to the crystal structure,

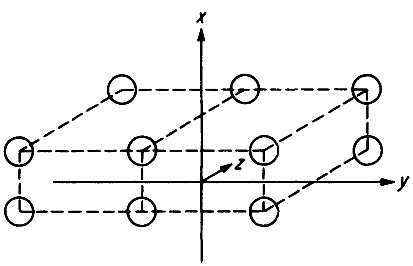

FIGURE 2.4
A noncubic crystal exhibiting anisotropic effects.

the relation between **D** and **E** would become

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z$$
$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z$$
$$D_z = \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z$$

or in matrix form,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{2.30}$$

Only for a particular orientation of the coordinate system does (2.30) reduce to (2.29). This particular orientation defines the principal axis of the medium. For anisotropic media the permittivity is referred to as a tensor permittivity (a tensor of rank 2 may be represented by a matrix). For the most part the materials dealt with in this text are isotropic. Nevertheless, an awareness of the existence of anisotropic media and of the nature of the constitutive relations for such media is important.

For the magnetic case, **H** is defined by the constitutive relation

$$\mu_0\mathbf{H} = \mathbf{B} - \mu_0\mathbf{M} \tag{2.31}$$

where **M** is the magnetic dipole polarization per unit volume. For most materials (ferromagnetic materials excluded), **M** is linearly related to **B** and hence to **H**. By convention this is expressed by the equation

$$\mathbf{M} = \chi_m\mathbf{H} \tag{2.32}$$

where χ_m is called the magnetic susceptibility. Substituting (2.32) into (2.31) gives

$$\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{2.33}$$

where $\mu = \mu_0(1 + \chi_m)$ is called the permeability.

As in the electric case, damping forces cause μ to be a complex parameter with a negative imaginary part; that is, $\mu = \mu' - j\mu''$. Also, there are magnetic materials that are anisotropic; in particular, ferrites are anisotropic magnetic materials of great usefulness at microwave frequencies. These exhibit a tensor permeability of the following form:

$$[\mu] = \begin{bmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \tag{2.34}$$

when a static magnetic field is applied along the axis for which the permeability is μ_3. A discussion of ferrites and their uses is presented later; so further comments on their anisotropic properties is deferred until then.

In Sec. 2.1 care was taken to write Maxwell's equations in a form valid not only in vacuum but also in material media. Thus (2.13) and (2.18) are valid in general, but with the constitutive relations of this section replacing the free-space relations (2.2). Note, however, that it is not possible to write, in general, constitutive relations of the form $\mathscr{D} = \epsilon\mathscr{E}$, $\mathscr{B} = \mu\mathscr{H}$, when $\mathscr{D}$ and $\mathscr{E}$, and likewise $\mathscr{B}$ and $\mathscr{H}$, are not in time phase. For arbitrary time dependence we must write instead $\mathscr{D} = \epsilon_0\mathscr{E} + \mathscr{P}$, $\mathscr{B} = \mu_0(\mathscr{H} + \mathscr{M})$ and relate $\mathscr{P}$ and $\mathscr{M}$ to $\mathscr{E}$ and $\mathscr{H}$ through the dynamical equation of motion governing the polarization mechanism. This difficulty may be circumvented by using the phasor representation for which relations such as $\mathbf{D} = \epsilon\mathbf{E}$ are perfectly valid because the complex nature of ϵ accounts for the difference in time phase.† It should be pointed out, however, that for many materials used at frequencies up to and including microwaves, the losses are so small that $\mathscr{D}$ and $\mathscr{E}$, and also $\mathscr{H}$ and $\mathscr{B}$, are very nearly in time phase. In such cases constitutive relations such as $\mathscr{D} = \epsilon\mathscr{E}$, $\mathscr{B} = \mu\mathscr{H}$ apply with negligible error. Significant departure in time phase between $\mathscr{D}$ and $\mathscr{E}$ or $\mathscr{B}$ and $\mathscr{H}$ occurs only in the vicinity of a natural resonance frequency of the equation of motion for the polarization.

2.3 STATIC FIELDS

For electric and magnetic fields that are independent of time, the electric and magnetic fields are not coupled, and likewise the current and charge are not coupled. Putting all time derivatives equal to zero in (2.13) yields‡

$$\nabla \times \mathbf{E} = 0 \tag{2.35a}$$

$$\nabla \cdot \epsilon\mathbf{E} = \rho \tag{2.35b}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2.36a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.36b}$$

$$\nabla \cdot \mathbf{J} = 0 \tag{2.36c}$$

The last equation is the continuity equation for the special case $\partial\rho/\partial t = 0$.

The static electric field has zero curl, or circulation, and this means that the line integral of $\mathbf{E}$ around any arbitrary closed contour is zero. This property is just the condition that permits $\mathbf{E}$ to be derived from the gradient of a scalar potential function Φ; that is, since $\nabla \times \nabla\Phi$ is identically zero, we may put

$$\mathbf{E} = -\nabla\Phi \tag{2.37}$$

†The situation here is like that encountered in ac circuit analysis, where in phasor notation the voltage V equals the current I multiplied by the impedance Z; that is, $V = IZ$. An Ohm's law of this sort cannot be written for the physical voltage and current, for if $\mathscr{V} = \mathrm{Re}(Ve^{j\omega t}) = V\cos\omega t$, then $\mathscr{I} = \mathrm{Re}(Ie^{j\omega t}) = [V/(R^2 + X^2)^{1/2}]\cos(\omega t - \phi)$, where $\phi = \tan^{-1}(X/R)$. Clearly, $\mathscr{V}$ cannot be equated to $\mathscr{I}$ multiplied by a constant because of the difference in phase.

‡For static fields we are using boldface roman type to represent the physically real vector fields.

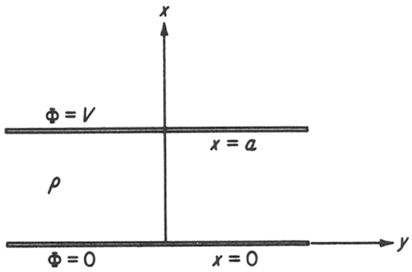

FIGURE 2.5
A simple potential problem.

Substituting (2.37) into (2.35*b*) and assuming that ϵ is a constant independent of the coordinates give

$$-\nabla \cdot \mathbf{E} = \nabla^2 \Phi = -\frac{\rho}{\epsilon} \tag{2.38}$$

This equation is known as Poisson's equation. When $\rho = 0$, Laplace's equation

$$\nabla^2 \Phi = 0 \tag{2.39}$$

is obtained. The basic field problem in electrostatics is to solve Poisson's or Laplace's equation for a potential function Φ that satisfies specified boundary conditions.

As a simple example consider two infinite conducting planes at $x = 0, a$, as in Fig. 2.5. Let charge be distributed with a density $\rho = \rho_0 x$ between the two plates.† It is required to find a Φ which is a solution of Poisson's equation and which equals zero on the plane $x = 0$ and V on the plane $x = a$. The potential will depend on x only; so (2.38) becomes

$$\frac{d^2\Phi}{dx^2} = -\rho_0 \frac{x}{\epsilon_0}$$

Integrating this equation twice gives $\Phi = -\rho_0 x^3/6\epsilon_0 + C_1 x + C_2$. Imposing the boundary conditions at $x = 0, a$ yields $0 = C_2$,

$$V = -\frac{\rho_0 a^3}{6\epsilon_0} + C_1 a + C_2$$

†The example is somewhat artificial since the assumed charge distribution is not a stable one; i.e., the electric field it produces would cause the charge distribution to change.

and hence $C_2 = 0$, $C_1 = V/a + \rho_0 a^2/6\epsilon_0$. The solution for Φ is thus

$$\Phi = -\frac{\rho_0 x^3}{6\epsilon_0} + \frac{\rho_0 a^2 x}{6\epsilon_0} + \frac{V}{a}x$$

The electric field between the two plates is

$$\mathbf{E} = -\nabla\Phi = -\mathbf{a}_x \frac{\partial \Phi}{\partial x} = \mathbf{a}_x \left(\frac{\rho_0 x^2}{2\epsilon_0} - \frac{\rho_0 a^2}{6\epsilon_0} - \frac{V}{a} \right)$$

The solution for the electrostatic field is greatly facilitated by introduction of the scalar potential Φ. For the same reason it is advantageous to introduce a potential function for the solution of magnetostatic problems. Since $\mathbf{B}$ always has zero divergence, it may be derived from the curl of a vector potential $\mathbf{A}$; that is,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.40}$$

This makes the divergence of $\mathbf{B}$ vanish identically because the divergence of the curl of a vector is identically zero. Using (2.40) in (2.36a) and assuming that μ is constant yields the equation

$$\nabla \times \mu\mathbf{H} = \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J}$$

A vector identity of use here is $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}$. The divergence of $\mathbf{A}$ may be placed equal to zero without affecting the value of $\mathbf{B}$ derived from the curl of $\mathbf{A}$, and hence the equation for $\mathbf{A}$ is

$$\nabla^2\mathbf{A} = -\mu\mathbf{J} \tag{2.41}$$

This equation is a vector Poisson's equation. In rectangular coordinates, (2.41) represents three scalar Poisson's equations, the first being

$$\nabla^2 A_x = -\mu J_x \tag{2.42}$$

In a curvilinear coordinate system, such as a cylindrical coordinate system, (2.41) cannot be written in such a simple component form. The reason is that, for example, $\nabla^2\mathbf{a}_r A_r$ does not equal $\mathbf{a}_r \nabla^2 A_r$ because, even though the unit vector $\mathbf{a}_r$ is of constant length, its orientation varies from point to point since it is always directed along the radius vector from the origin to the point under consideration. The evaluation of $\nabla^2\mathbf{A}$ in curvilinear coordinates is made by using the vector identity quoted above to give $\nabla^2\mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A}$. These latter operations are readily carried out.

The interest in static field solutions at microwave frequencies arises because the field distribution over a cross-sectional plane of a transmission line is a static field distribution and because static field solutions are good approximate solutions to the actual fields in the vicinity of obstacles that are small compared with the wavelength. The potential theory introduced above may be extended to the time-varying case also, and this is done in a following section.

2.4 WAVE EQUATION

For convenience, the two curl equations are repeated here:

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t} \tag{2.43a}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} \tag{2.43b}$$

where it is assumed for the present that the current density $\mathscr{J}$ is zero in the region of interest. These equations, together with the assumed constitutive relations $\mathscr{D} = \epsilon\mathscr{E}$, $\mathscr{B} = \mu\mathscr{H}$, may be combined to obtain a separate equation for each field. The curl of (2.43a) is

$$\nabla \times \nabla \times \mathscr{E} = -\frac{\partial(\nabla \times \mathscr{B})}{\partial t} = -\mu\frac{\partial(\nabla \times \mathscr{H})}{\partial t}$$

Using (2.43b) and expanding $\nabla \times \nabla \times \mathscr{E}$ now yields

$$\nabla\nabla \cdot \mathscr{E} - \nabla^2\mathscr{E} = -\mu\epsilon\frac{\partial^2\mathscr{E}}{\partial t^2}$$

Since ρ is assumed zero and ϵ is taken as a constant, $\nabla \cdot \mathscr{E} = 0$, and we obtain

$$\nabla^2\mathscr{E} - \mu\epsilon\frac{\partial^2\mathscr{E}}{\partial t^2} = 0 \tag{2.44}$$

which is a three-dimensional wave equation. The velocity of propagation v is equal to $(\mu\epsilon)^{-1/2}$. In free space v is equal to the velocity of light c. To illustrate the nature of the solutions of (2.44), consider a case where $\mathscr{E}$ has only an x component and depends only on the z coordinate. In this instance

$$\frac{\partial^2\mathscr{E}_x}{\partial z^2} - \mu\epsilon\frac{\partial^2\mathscr{E}_x}{\partial t^2} = 0$$

Any function of the form $f(z - vt)$ is a solution of this equation since

$$\frac{\partial^2 f}{\partial z^2} = f'' \qquad \frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial(vt)^2} = v^2 f''$$

and hence

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$$

This solution is illustrated in Fig. 2.6 and clearly represents a disturbance propagating in the positive z direction with velocity v. An equally valid solution is $f(z + vt)$ and represents a disturbance propagating in the negative z direction.

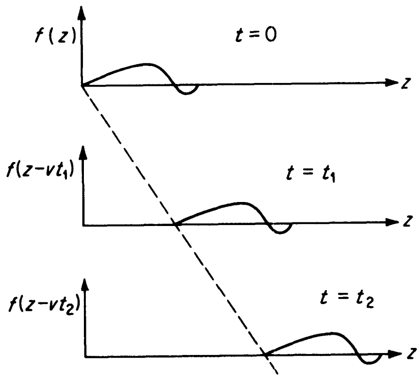

FIGURE 2.6
Propagation of a disturbance $f(z - vt)$.

By eliminating the electric field, it is readily found that the magnetic field $\mathscr{H}$ also satisfies the wave equation (2.44). In practice, however, we solve the wave equation for either $\mathscr{E}$ or $\mathscr{H}$ and then derive the other field by using the appropriate curl equation. When constitutive relations such as $\mathscr{D} = \epsilon\mathscr{E}$ and $\mathscr{B} = \mu\mathscr{H}$ cannot be written, the polarization vectors $\mathscr{P}$ and $\mathscr{M}$ must be exhibited explicitly in Maxwell's equations. Wave equations for $\mathscr{E}$ and $\mathscr{H}$ may still be derived, but $\mathscr{P}$ and $\mathscr{M}$ will now enter as equivalent sources for the field (which they actually are). The derivation is left as a problem at the end of this chapter.

For harmonic time dependence, the equation obtained in place of (2.44) is

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{2.45}$$

where $k^2 = \omega^2\mu\epsilon$. This equation is referred to as the Helmholtz equation, or reduced wave equation. The constant k is called the wave number and may be expressed in the form

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v} = 2\pi\frac{f}{v} = \frac{2\pi}{\lambda} \tag{2.46}$$

where the wavelength λ is equal to v/f. In free space the wave number will be written as k_0, and is equal to $\omega\sqrt{\mu_0\epsilon_0} = 2\pi/\lambda_0$. The magnetic field $\mathbf{H}$, as may be surmised, satisfies the same reduced wave equation.

In a medium with finite conductivity σ, a conduction current $\mathscr{J} = \sigma\mathscr{E}$ will exist, and this results in energy loss because of Joule heating. The wave equation in media of this type has a damping term proportional to σ and the first time derivative of the field. In metals, excluding ferromagnetic materials, the permittivity and permeability are essentially equal to their free-space values, at least for frequencies up to and including the microwave range. Thus Maxwell's curl equations become

$$\nabla \times \mathscr{E} = -\mu_0\frac{\partial\mathscr{H}}{\partial t} \qquad \nabla \times \mathscr{H} = \epsilon_0\frac{\partial\mathscr{E}}{\partial t} + \sigma\mathscr{E}$$

Elimination of $\mathscr{H}$ in the same manner as before leads to the following wave equation for $\mathscr{E}$:

$$\nabla^2\mathscr{E} - \mu_0\sigma\frac{\partial\mathscr{E}}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\mathscr{E}}{\partial t^2} = 0 \tag{2.47}$$

The magnetic field $\mathscr{H}$ also satisfies this equation. For the time-harmonic case damping effects enter in through the complex nature of ϵ and μ, and hence the wave number k. It should be recalled here that, as shown by (2.27), a finite conductivity σ is equivalent to an imaginary term in the permittivity ϵ. In the present case the equivalent permittivity is $\epsilon = \epsilon_0 - j\sigma/\omega$ and the Helmholtz equation is

$$\nabla^2\mathbf{E} + \omega^2\mu_0\epsilon_0\left(1 - j\frac{\sigma}{\omega\epsilon_0}\right)\mathbf{E} = 0 \tag{2.48}$$

In metals the conduction current $\sigma\mathbf{E}$ is generally very much larger than the displacement current $\omega\epsilon_0\mathbf{E}$, so that the latter may be neglected. For example, σ is equal to 5.8×10^7 S/m for copper, and at a frequency of 10^{10} Hz, $\omega\epsilon_0 = 0.55$, which is much smaller than σ. Only for frequencies in the optical range will the two become comparable. Thus (2.47) may be simplified to

$$\nabla^2\mathscr{E} - \mu_0\sigma\frac{\partial\mathscr{E}}{\partial t} = 0 \tag{2.49}$$

and (2.48) reduces to

$$\nabla^2\mathbf{E} - j\omega\mu_0\sigma\mathbf{E} = 0 \tag{2.50}$$

Equation (2.49) is a diffusion equation similar to that which governs the flow of heat in a thermal conductor.

2.5 ENERGY AND POWER

When currents exist in conductors as a result of the application of a suitable potential source, energy is expended by the source in maintaining the currents. The energy supplied by the source is stored in the electric and magnetic fields set up by the currents or propagated (radiated) away in the form of an electromagnetic wave. Under steady-state sinusoidal time-varying conditions, the time-average energy stored in the electric field is

$$W_e = \operatorname{Re}\frac{1}{4}\int_V \mathbf{E}\cdot\mathbf{D}^*\,dV = \frac{1}{4}\int_V \epsilon'\mathbf{E}\cdot\mathbf{E}^*\,dV \tag{2.51a}$$

If ϵ is a constant and real, (2.51a) becomes

$$W_e = \frac{\epsilon}{4}\int_V \mathbf{E}\cdot\mathbf{E}^*\,dV \tag{2.51b}$$

The time-average energy stored in the magnetic field is given by

$$W_m = \operatorname{Re} \frac{1}{4} \int_V \mathbf{H}^* \cdot \mathbf{B} \, dV = \frac{1}{4} \int_V \mu' \mathbf{H} \cdot \mathbf{H}^* \, dV \tag{2.52a}$$

which, for μ real and constant, becomes

$$W_m = \frac{\mu}{4} \int_V \mathbf{H} \cdot \mathbf{H}^* \, dV \tag{2.52b}$$

These expressions for W_e and W_m are valid only for nondispersive media, i.e., media for which ϵ and μ can be considered independent of ω in the vicinity of the angular frequency ω with which the fields vary. In general, when the losses are small, so that $\epsilon'' \ll \epsilon'$ and $\mu'' \ll \mu'$, we have

$$W_e = \frac{1}{4} \int_V \mathbf{E} \cdot \mathbf{E}^* \frac{\partial \omega \epsilon'}{\partial \omega} \, dV \tag{2.53a}$$

$$W_m = \frac{1}{4} \int_V \mathbf{H} \cdot \mathbf{H}^* \frac{\partial \omega \mu'}{\partial \omega} \, dV \tag{2.53b}$$

for the time-average stored electric and magnetic energy.

The above equations for the time-average energy in a dispersive medium may be established by considering a classical model of the polarization mechanism similar to that discussed in Sec. 2.2. In a unit volume let the effective oscillating charge of the dipole distribution be $-q$ with an effective mass m. Let the damping force be equal to $m\nu$ times the velocity of the charge. This damping force takes account of collision effects and loss of energy by radiation from the oscillating charge. The equation of motion for the polarization charge displacement u is

$$m \frac{d^2 u}{dt^2} + m\nu \frac{du}{dt} + ku = -q\mathscr{E}$$

where u is parallel to the direction of the field $\mathscr{E}$. In this equation k is the elastic constant giving rise to the restoring force. This constant arises from the Coulomb forces acting on the displaced charge, and hence is of electrical origin. The dipole polarization $\mathscr{P}$ is $-qu$, and the polarization current $\mathscr{J}_p = d\mathscr{P}/dt$. Introducing the polarization current into the equation of motion gives

$$\frac{m}{q^2} \frac{d\mathscr{J}_p}{dt} + \frac{m\nu}{q^2} \mathscr{J}_p + \frac{k}{q^2} \int^t \mathscr{J}_p \, dt = \mathscr{E}$$

This equation is formally the same as that which describes the current in a series LCR circuit with an applied voltage $\mathscr{V}$ equal to $\mathscr{E}$ and with

$$L = \frac{m}{q^2} \qquad R = \frac{m\nu}{q^2} \qquad C = \frac{q^2}{k}$$

An equivalent circuit describing the polarization is illustrated in Fig. 2.7. If

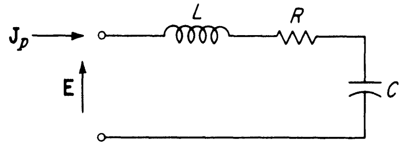

FIGURE 2.7
Equivalent circuit for polarization current.

a time dependence $e^{j\omega t}$ is assumed and phasor notation is used,

$$J_p = EY = E\frac{R - jX}{R^2 + X^2}$$

where Y is the input admittance and $X = \omega L - 1/\omega C$. Since $P = \epsilon_0 \chi_e E$ and $J_p = j\omega P$, we see that

$$\omega\epsilon_0\chi_e = \omega\epsilon_0(\chi_e' - j\chi_e'') = -jY = \frac{-X - jR}{R^2 + X^2}$$

and hence

$$\omega\epsilon_0\chi_e' = \frac{-X}{R^2 + X^2} \tag{2.54a}$$

$$\omega\epsilon_0\chi_e'' = \frac{R}{R^2 + X^2} \tag{2.54b}$$

The time-average power loss associated with the polarization is the same as the power loss in R in the equivalent circuit. This is given by

$$P_l = \frac{1}{2}EE^*\frac{R}{R^2 + X^2} = \frac{1}{2}EE^*\omega\epsilon_0\chi_e'' \tag{2.55}$$

per unit volume. This equation shows that $\omega\epsilon_0\chi_e'' = \omega\epsilon''$ is an equivalent conductance. The time-average energy stored in the system is of two forms. First there is the kinetic energy of motion, that is, $\frac{1}{2}m(du/dt)^2$ averaged over a cycle, and this is equal to the magnetic energy stored in the inductor in the equivalent circuit. This time-average kinetic energy per unit volume is given by

$$U_m = \frac{1}{4}LJ_pJ_p^* = \frac{1}{4}EE^*\frac{L}{R^2 + X^2} \tag{2.56a}$$

The second form of stored energy is the potential energy associated with the charge displacement. The time-average value of this energy is equal to the time-average electric energy stored in the capacitor C in the equivalent circuit, and is given by

$$U_e = \frac{1}{4}EE^*\frac{1}{\omega^2 C(R^2 + X^2)} \tag{2.56b}$$

The total time-average energy stored per unit volume is $U = U_m + U_e$. Note that U is not given by $\frac{1}{4}EE^*\epsilon_0\chi'_e$. The latter expression gives

$$\frac{1}{4}EE^*\epsilon_0\chi'_e = \frac{1}{4}EE^*\frac{-X}{\omega(R^2+X^2)}$$
$$= \frac{1}{4}EE^*\frac{1/\omega^2C - L}{R^2+X^2} = U_e - U_m$$

or the difference between the potential and kinetic energy stored.

To obtain an expression for the total stored energy, note that

$$\frac{d}{d\omega}\left(\frac{X}{R^2+X^2}\right) = \frac{L+1/\omega^2C}{R^2+X^2}\left(1-\frac{2X^2}{R^2+X^2}\right)$$

For a low-loss system, $R^2 \ll X^2$, and we then have $1 - 2X^2/(R^2+X^2) \approx -1$; so

$$\frac{d}{d\omega}\left(\frac{-X}{R^2+X^2}\right) = \frac{d}{d\omega}(\omega\epsilon_0\chi'_e) \approx \frac{L+1/\omega^2C}{R^2+X^2}$$

Multiplying this expression by $\frac{1}{4}EE^*$ now gives the total time-average energy stored, as comparison with (2.56a) and (2.56b) shows. Thus the final expression for the time-average electric energy stored in a volume V is given by the volume integral of $U = U_e + U_m$ plus the free-space energy density $\epsilon_0(\mathbf{E}\cdot\mathbf{E}^*)/4$ and is

$$W_e = \int_V\left(U + \frac{\epsilon_0}{4}\mathbf{E}\cdot\mathbf{E}^*\right)dV$$
$$= \int_V\frac{\mathbf{E}\cdot\mathbf{E}}{4}\left(\epsilon_0 + \frac{\partial\omega\epsilon_0\chi'_e}{\partial\omega}\right)dV$$
$$= \frac{1}{4}\int_V\mathbf{E}\cdot\mathbf{E}^*\frac{\partial\omega\epsilon'}{\partial\omega}\,dV$$

since $\epsilon' = \epsilon_0(1+\chi'_e)$. This equation is the result given earlier by (2.53a).

A similar type of model may be used to establish (2.53b) for the average stored magnetic energy. It should be pointed out that under time-varying conditions the average stored energy associated with either electric or magnetic polarization includes a kinetic-energy term. This term is negligible at low frequencies and also when ϵ' and μ' are essentially independent of ω for the range of ω of interest. When this energy is not negligible, the modified expressions for stored energy must be used.

Although (2.53) is more general than (2.51) and (2.52), we shall, in the majority of instances, use the latter equations for the stored energy. We thereby tacitly assume that we are dealing with material that is nondispersive or very nearly so.

The time-average power transmitted across a closed surface S is given by the integral of the real part of one-half of the normal component of the complex Poynting vector $\mathbf{E} \times \mathbf{H}^*$; that is,

$$P = \text{Re}\,\frac{1}{2}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} \tag{2.57}$$

The above results are obtained from the interpretation of the complex Poynting vector theorem, which may be derived from Maxwell's equations as follows: If the divergence of $\mathbf{E} \times \mathbf{H}^*$, that is, $\nabla \cdot \mathbf{E} \times \mathbf{H}^*$, is expanded, we obtain

$$\nabla \cdot \mathbf{E} \times \mathbf{H}^* = (\nabla \times \mathbf{E}) \cdot \mathbf{H}^* - (\nabla \times \mathbf{H}^*) \cdot \mathbf{E}$$

From Maxwell's equations $\nabla \times \mathbf{E} = -j\omega\mathbf{B}$ and $\nabla \times \mathbf{H}^* = -j\omega\mathbf{D}^* + \mathbf{J}^*$, and hence

$$\nabla \cdot \mathbf{E} \times \mathbf{H}^* = -j\omega\mathbf{B} \cdot \mathbf{H}^* + j\omega\mathbf{D}^* \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{J}^*$$

The integration of this equation throughout a volume V bounded by a closed surface S gives the complex Poynting vector theorem; i.e.,

$$\begin{aligned} \frac{1}{2}\int_V \nabla \cdot \mathbf{E} \times \mathbf{H}^*\, dV &= \frac{1}{2}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot d\mathbf{S} \\ &= -j\frac{\omega}{2}\int_V (\mathbf{B} \cdot \mathbf{H}^* - \mathbf{E} \cdot \mathbf{D}^*)\, dV - \frac{1}{2}\int_V \mathbf{E} \cdot \mathbf{J}^*\, dV \end{aligned} \tag{2.58a}$$

where the divergence theorem has been used on the left-hand side integral. The above result may be rewritten as

$$\begin{aligned} \frac{1}{2}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot (-d\mathbf{S}) &= 2j\omega\int_V \left(\frac{\mathbf{B} \cdot \mathbf{H}^*}{4} - \frac{\mathbf{E} \cdot \mathbf{D}^*}{4}\right) dV \\ &\quad + \frac{1}{2}\int_V \mathbf{E} \cdot \mathbf{J}^*\, dV \end{aligned} \tag{2.58b}$$

where $-d\mathbf{S}$ is a vector element of area directed into the volume V. If the medium in V is characterized by parameters $\epsilon = \epsilon' - j\epsilon''$, $\mu = \mu' - j\mu''$, and conductivity σ, the real and imaginary parts of (2.58) may be equated to give

$$\begin{aligned} \text{Re}\,\frac{1}{2}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot (-d\mathbf{S}) &= \frac{\omega}{2}\int_V (\mu''\mathbf{H} \cdot \mathbf{H}^* + \epsilon''\mathbf{E} \cdot \mathbf{E}^*)\, dV \\ &\quad + \frac{1}{2}\int_V \sigma\mathbf{E} \cdot \mathbf{E}^*\, dV \end{aligned} \tag{2.59a}$$

$$\text{Im}\,\frac{1}{2}\oint_S \mathbf{E} \times \mathbf{H}^* \cdot (-d\mathbf{S}) = 2\omega\int_V \left(\mu'\frac{\mathbf{H} \cdot \mathbf{H}^*}{4} - \epsilon'\frac{\mathbf{E} \cdot \mathbf{E}^*}{4}\right) dV \tag{2.59b}$$

Equation (2.59*a*) is interpreted to state that the real electromagnetic power transmitted through the closed surface S into V is equal to the power loss produced by conduction current $\sigma\mathbf{E}$, resulting in Joule heating plus the power loss resulting from polarization damping forces. Note that $\omega\epsilon''$ could be interpreted as an equivalent conductance, as pointed out in Sec. 2.2. This equation also shows that μ'' and ϵ'' must be positive in order to represent energy loss, and hence the imaginary parts of ϵ and μ must be negative. Equation (2.59*b*) states that the imaginary part of the complex rate of energy flow into V is equal to 2ω times the net reactive energy $W_m - W_e$ stored in the magnetic and electric fields in V. The complex Poynting vector theorem is essentially an energy-balance equation.

A result analogous to the above may be derived for a conventional network, and serves to demonstrate the validity of the interpretation of (2.58). Consider a simple series RLC circuit as in Fig. 2.7. If the current in the circuit is I and the applied voltage is V, the complex input power is given by

$$\frac{1}{2}VI^* = \frac{1}{2}ZII^* = \frac{1}{2}II^*\left(R + j\omega L - \frac{j}{\omega C}\right)$$

The time-average power loss in R, magnetic energy stored in the field around L, and electric energy stored in the field associated with C are given, respectively, by

$$P_l = \frac{1}{2}RII^* \qquad W_m = \frac{1}{4}LII^* \qquad W_e = \frac{1}{4}\frac{II^*}{\omega^2 C}$$

since the voltage across C is $I/\omega C$. Hence

$$\frac{1}{2}VI^* = \frac{1}{2}ZII^* = P_l + 2j\omega(W_m - W_e)$$

which has the same interpretation as (2.58). This equation may also be solved for the impedance Z to give

$$Z = \frac{P_l + 2j\omega(W_m - W_e)}{\frac{1}{2}II^*} \tag{2.60}$$

and provides a general definition of the impedance of a network in terms of the associated power loss and stored reactive energy. The factor $\frac{1}{2}II^*$ in the denominator serves as a normalization factor, and is required in order to make Z independent of the magnitude of the current at the input to the network.

In the case of a general time-varying field, an expansion of $\nabla \cdot \mathscr{E} \times \mathscr{H}$ and substitution from Maxwell's equations (2.13) lead to the following Poynting vector theorem for general time-varying fields:

$$\oint_S \mathscr{E} \times \mathscr{H} \cdot (-d\mathbf{S}) = \int_V \left(\mu_0 \mathscr{H} \cdot \frac{\partial \mathscr{H}}{\partial t} + \mu_0 \mathscr{H} \cdot \frac{\partial \mathscr{M}}{\partial t} \right.$$
$$\left. + \epsilon_0 \mathscr{E} \cdot \frac{\partial \mathscr{E}}{\partial t} + \mathscr{E} \cdot \frac{\partial \mathscr{P}}{\partial t} + \mathscr{E} \cdot \mathscr{J} \right) dV$$

Since $\mathscr{H} \cdot \partial\mathscr{H}/\partial t = \frac{1}{2}\partial(\mathscr{H} \cdot \mathscr{H})/\partial t$, etc., and the electric and magnetic polarization currents are $\mathscr{J}_p = \partial\mathscr{P}/\partial t$, $\mathscr{J}_m = \mu_0(\partial\mathscr{M}/\partial t)$, we have

$$\oint_S \mathscr{E} \times \mathscr{H} \cdot (-d\mathbf{S}) = \frac{\partial}{\partial t}\int_V \left(\frac{\mu_0 \mathscr{H} \cdot \mathscr{H}}{2} + \frac{\epsilon_0 \mathscr{E} \cdot \mathscr{E}}{2} \right) dV$$

$$+ \int_V [\mathscr{E} \cdot (\mathscr{J} + \mathscr{J}_p) + \mathscr{H} \cdot \mathscr{J}_m] \, dV \qquad (2.61)$$

where $-d\mathbf{S}$ is an element of surface area directed into V. This equation states that the rate of energy flow into V is equal to the time rate of change of the free-space field energy stored in V plus the rate of energy dissipation in Joule heating arising from the conduction current $\mathscr{J}$ and, in addition, the instantaneous rate of energy supplied in maintaining the polarization currents. If $\mathscr{M}$ and $\mathscr{H}$, and also $\mathscr{P}$ and $\mathscr{E}$, are in phase, there is no energy loss associated with the polarization currents. If these quantities are not in phase, some energy dissipation takes place, leading to increased heating of the material.

If the susceptibilities χ_e and χ_m can be considered as constants, so that $\partial\mathscr{P}/\partial t = \epsilon_0\chi_e(\partial\mathscr{E}/\partial t)$ and $\partial\mathscr{M}/\partial t = \chi_m(\partial\mathscr{H}/\partial t)$, then (2.61) becomes

$$\oint_S \mathscr{E} \times \mathscr{H} \cdot (-d\mathbf{S}) = \frac{\partial}{\partial t}\int_V \left(\frac{\mathscr{H} \cdot \mathscr{B}}{2} + \frac{\mathscr{E} \cdot \mathscr{D}}{2} \right) dV + \int_V \mathscr{E} \cdot \mathscr{J} \, dV \quad (2.62)$$

which is the usual form of the Poynting vector theorem. The first term on the right is now interpreted as the instantaneous rate of change of the total electric and magnetic energy stored in the volume V.

The susceptibilities can usually be considered as true constants whenever the inertial and damping forces are small compared with the elastic restoring force in the dynamical equation describing the polarization. For example, with reference to (2.54a), this is the case when k is much greater than $\omega m \nu$ or $\omega^2 m$, that is, when $1/\omega C$ is large compared with ωL and R, so that

$$\epsilon_0 \chi_e' \approx C = \frac{q^2}{k}$$

2.6 BOUNDARY CONDITIONS

In order to find the proper and unique solutions to Maxwell's equations for situations of practical interest (these always involve material bodies with boundaries), a knowledge of the behavior of the electromagnetic field at the boundary separating material bodies with different electrical properties is required. From a mathematical point of view, the solution of a partial differential equation, such as a wave equation, in a region V is not unique unless boundary conditions are specified, i.e., the behavior of the field on the boundary of V. Boundary conditions play the same role in the solution of

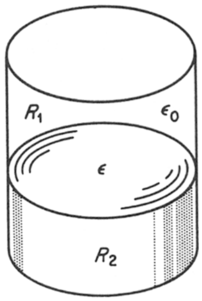

FIGURE 2.8
A cylindrical cavity partially filled with a dielectric medium.

partial differential equations that initial conditions play in the solution of the differential equations that govern the behavior of electric circuits.

As an example, consider the problem of finding a solution to Maxwell's equations inside a cylindrical cavity partially filled with a dielectric medium of permittivity ϵ, as in Fig. 2.8. In practice, the solution is obtained by finding general solutions valid in the two regions labeled R_1 and R_2. These general solutions must satisfy prescribed conditions on the metallic boundaries and in addition contain arbitrary amplitude constants that can be determined only from a knowledge of the boundary conditions to be applied at the air-dielectric boundary separating regions R_1 and R_2.

The integral form of Maxwell's equations provides the most convenient formulation in order to deduce the required boundary conditions. Consider two media with parameters ϵ_1, μ_1 and ϵ_2, μ_2, as in Fig. 2.9*a*. If there is no surface charge on the boundary, which is the usual case for nonconducting media, the integral of the displacement flux over the surface of the small "coin-shaped" volume centered on the boundary as in Fig. 2.9*b* gives, in the limit as h tends to zero,

$$\lim_{h \to 0} \oint_S \mathbf{D} \cdot d\mathbf{S} = D_{2n}\,\Delta S - D_{1n}\,\Delta S = 0$$

or

$$D_{2n} = D_{1n} = \mathbf{n} \cdot \mathbf{D}_2 = \mathbf{n} \cdot \mathbf{D}_1 \tag{2.63}$$

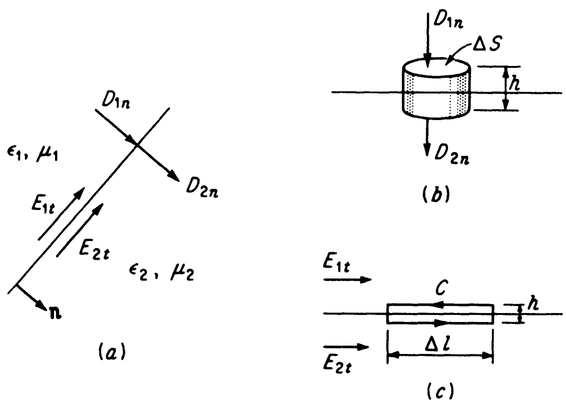

FIGURE 2.9
Boundary between two different media.

where n denotes the normal component. The limit $h \to 0$ is taken so that the flux through the sides of the coin-shaped region vanishes. Equation (2.63) simply states that the displacement flux lines are continuous in the direction normal to the boundary. A similar result clearly must hold also for the magnetic flux lines since $\nabla \cdot \mathbf{B} = 0$, and hence, by analogy,

$$\mathbf{n} \cdot \mathbf{B}_2 = \mathbf{n} \cdot \mathbf{B}_1 \tag{2.64}$$

To obtain boundary conditions on the tangential components of the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$, the circulation integrals for $\mathbf{E}$ and $\mathbf{H}$ are used. If for the contour C in Fig. 2.9c, the width h is made to approach zero, the magnetic flux flowing through this contour vanishes and

$$\begin{aligned}\lim_{h\to 0} \oint_C \mathbf{E} \cdot d\mathbf{l} &= \lim_{h \to 0} -j\omega \int_S \mathbf{B} \cdot d\mathbf{S} = 0 \\ &= E_{2t}\,\Delta l - E_{1t}\,\Delta l\end{aligned}$$

or

$$E_{1t} = E_{2t} \tag{2.65}$$

For the same contour C the total displacement current directed through the contour vanishes as $h \to 0$, so that

$$\begin{aligned}\lim_{h\to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} &= \lim_{h \to 0} \left(j\omega \int_S \mathbf{D} \cdot d\mathbf{S} \right) = 0 \\ &= (H_{2t} - H_{1t})\,\Delta l\end{aligned}$$

or

$$H_{2t} = H_{1t} \tag{2.66}$$

where t denotes the components tangential to the boundary surface. These latter relations state that the components of $\mathbf{E}$ and $\mathbf{H}$ tangent to the boundary are continuous across the boundary; i.e., the tangential components on adjacent sides of the boundary are equal at the boundary surface.

For the boundary conditions at the surface separating a good conductor (any metal) and free space or air, some simplification is possible. As shown in a later section, the electromagnetic field can penetrate into a conductor only a minute distance at microwave frequencies. The field amplitude decays exponentially from its surface value according to e^{-u/δ_s}, where u is the normal distance into the conductor measured from the surface, and δ_s is called the skin depth. The skin depth is given by

$$\delta_s = \left(\frac{2}{\omega \mu \sigma} \right)^{1/2} \tag{2.67}$$

For copper ($\sigma = 5.8 \times 10^7$ S/m) at a frequency of 10^{10} Hz, the skin depth is 6.6×10^{-5} cm, truly a very small distance. Likewise, the current $\mathbf{J} = \sigma\mathbf{E}$ is concentrated near the surface. As the conductivity is made to approach infinity, δ_s approaches zero and the current is squeezed into a narrower and narrower region and in the limit $\sigma \to \infty$ becomes a true surface current. Since the skin depth is so small at microwave frequencies for metals, the approximation of infinite conductivity may be made with negligible error (an

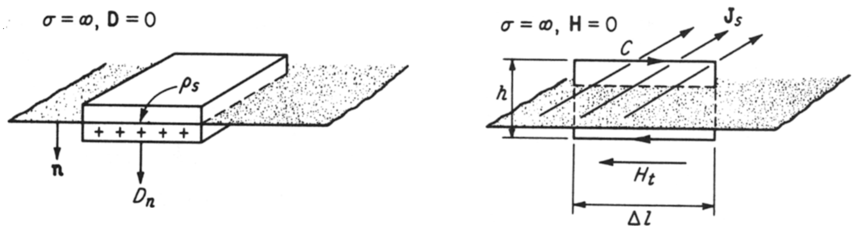

FIGURE 2.10
Boundary of a perfect conductor.

exception is when attenuation is to be calculated, since then infinite conductivity implies no loss). For infinite conductivity the field in the conductor must be zero. Since the flux lines of **B** are continuous and likewise since the tangential component of **E** is continuous across the boundary, it is necessary that

$$\mathbf{n} \cdot \mathbf{B} = 0 \tag{2.68a}$$

$$\mathbf{E}_t = \mathbf{n} \times \mathbf{E} = 0 \tag{2.68b}$$

at the surface of a perfect conductor. This same argument cannot be applied to the normal component of **D** and the tangential component of **H** because, as noted above, a surface current $\mathbf{J}_s$ will exist on the surface in the limit $\sigma \to \infty$. Applying Maxwell's equation

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = j\omega \int \mathbf{D} \cdot d\mathbf{S} + \int \mathbf{J} \cdot d\mathbf{S}$$

to the contour C illustrated in Fig. 2.10 gives

$$\begin{aligned}\lim_{h \to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} = H_t \,\Delta l &= \lim_{h \to 0} \int j\omega \mathbf{D} \cdot d\mathbf{S} + \lim_{h \to 0} \int \mathbf{J} \cdot d\mathbf{S} \\ &= \lim_{h \to 0} hJ\,\Delta l = J_s \,\Delta l\end{aligned}$$

or in vector form,

$$\mathbf{n} \times \mathbf{H} = \mathbf{J}_s \tag{2.68c}$$

Note that the field in the conductor goes to zero, that the total displacement current through C vanishes as $h \to 0$, but that hJ tends to the limiting value J_s as the conductivity is made infinite and h is made to approach zero. Associated with the surface current is a charge of density ρ_s on which the normal displacement flux lines terminate. Hence, at the surface of a perfect conductor,

$$\mathbf{n} \cdot \mathbf{D} = D_n = \rho_s \tag{2.68d}$$

When it is desired to take into account the large but finite conductivity (as would be the case in attenuation calculations), an impedance boundary condition may be used with little error. The metallic surface exhibits a surface impedance Z_m, with equal resistive and inductive parts, given by

$$Z_m = \frac{1+j}{\sigma\delta_s} \tag{2.69}$$

At the surface a surface current exists, and the relation between this and the electric field tangent to the surface is

$$\mathbf{E}_t = Z_m \mathbf{J}_s \tag{2.70}$$

Note that the tangential electric field cannot be zero for finite conductivity, although it may be very small. Now $\mathbf{n} \times \mathbf{H} = \mathbf{J}_s$, so that

$$\mathbf{E}_t = Z_m \mathbf{J}_s = Z_m \mathbf{n} \times \mathbf{H} \tag{2.71}$$

From (2.69) it is seen that the resistive part of the surface impedance is equal to the dc resistance per square of a unit square of metal of thickness δ_s. In a later section the above results are verified; so further comments are reserved until then.

In practice, it suffices to make the tangential components of the fields satisfy the proper boundary conditions since, when they do, the normal components of the fields automatically satisfy their appropriate boundary conditions. The reason is that when the fields are a solution of Maxwell's equations, not all the components of the field are independent. For example, when the tangential part of the electric field is continuous across a boundary, the derivatives of the tangential component of electric field with respect to coordinates on the boundary surface are also continuous. Thus the curl of the electric field normal to the surface is continuous, and this implies continuity of the normal component of $\mathbf{B}$. More specifically, if the xy plane is the boundary surface and E_x, E_y are continuous, then $\partial E_x/\partial x$, $\partial E_x/\partial y$, $\partial E_y/\partial x$, and $\partial E_y/\partial y$ are also continuous. Hence $-j\omega B_z = \partial E_y/\partial x - \partial E_x/\partial y$ is continuous. For the same reasons continuity of the tangential components of $\mathbf{H}$ ensures the continuity of the normal component of $\mathbf{D}$ across a boundary.

In addition to the boundary conditions given above, a boundary condition must be imposed on the field solutions at the edge of a conducting body such as a wedge. The edge condition requires that the energy stored in the field in the vicinity of an edge of a conducting body be finite. This limits the maximum rate at which the field intensities can increase as the edge is approached.† A detailed analysis shows that at the edge of a two-dimensional perfectly conducting wedge with an internal angle ϕ, the field components normal to the edge must not increase any faster than r^{α}, where r is

†J. Meixner, The Behavior of Electromagnetic Fields at Edges, *N.Y. Univ. Inst. Math. Sci. Res. Rept.*, vol. EM-72, December, 1954. The theory is also discussed in R. E. Collin, "Field Theory of Guided Waves," chap. 1, IEEE Press, Piscataway, N.J., 1991, revised edition.

the perpendicular radial distance away from the edge and

$$\alpha = \frac{n\pi}{2\pi - \phi} - 1$$

where the integer n must be chosen so that α is greater than or equal to $-\frac{1}{2}$ at least.

When solving for fields in an infinite region of space, the behavior of the field at infinity must also be specified. This boundary condition is called a radiation condition, and requires that the field at infinity be a wave propagating a finite amount of energy outward, or else that the field vanish so fast that the energy stored in the field and the energy flow at infinity are zero.

2.7 PLANE WAVES

In this section and the two following ones we shall introduce wave solutions by considering plane waves propagating in free space and reflection of a plane wave from a boundary separating free space and a dielectric, or conducting, medium. The latter problem will serve to derive the boundary conditions given by (2.68) to (2.71) in the preceding section.

Plane Waves in Free Space

The electric field is a solution of the Helmholtz equation

$$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = \frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} + \frac{\partial^2\mathbf{E}}{\partial z^2} + k_0^2\mathbf{E} = 0$$

This vector equation holds for each component of $\mathbf{E}$, so that

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} + k_0^2 E_i = 0 \qquad i = x, y, z \tag{2.72}$$

The standard procedure for solving a partial differential equation is the method of *separation of variables*. However, this method does not work for all types of partial differential equations in all various coordinate systems, and when it does not work, a solution is very difficult, if not impossible, to obtain. For the Helmholtz equation the method of separation of variables does work in such common coordinate systems as rectangular, cylindrical, and spherical. Hence this method suffices for the class of problems discussed in this text. The basic procedure is to assume for the solution a product of functions each of which is a function of one coordinate variable only. Substitution of this solution into the partial differential equation then separates the partial differential equation into three ordinary differential equations which may be solved by standard means.

In the present case let $E_x = f(x)g(y)h(z)$. Substituting this expression into (2.72) gives

$$ghf'' + fhg'' + fgh'' + k_0^2 fgh = 0$$

where the double prime denotes the second derivative. Dividing this equation by fgh gives

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} + k_0^2 = 0 \tag{2.73}$$

Each of the first three terms in (2.73), such as f''/f, is a function of a single independent variable only, and hence the sum of these terms can equal a constant $-k_0^2$ only if each term is constant. Thus (2.73) separates into three equations:

$$\frac{f''}{f} = -k_x^2 \qquad \frac{g''}{g} = -k_y^2 \qquad \frac{h''}{h} = -k_z^2$$

or

$$\frac{d^2f}{dx^2} + k_x^2 f = 0 \qquad \frac{d^2g}{dy^2} + k_y^2 g = 0 \qquad \frac{d^2h}{dz^2} + k_z^2 h = 0 \tag{2.74}$$

where k_x^2, k_y^2, k_z^2 are called separation constants. The only restriction so far on k_x^2, k_y^2, k_z^2 is that their sum must equal k_0^2, that is,

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 \tag{2.75}$$

so that (2.73) will be satisfied.

Equations (2.74) are simple-harmonic differential equations with exponential solutions of the form $e^{\pm jk_x x}, e^{\pm jk_y y}, e^{\pm jk_z z}$. As one suitable solution for E_x we may therefore choose

$$E_x = Ae^{-jk_x x - jk_y y - jk_z z} \tag{2.76}$$

where A is an amplitude factor. This solution is interpreted as the x component of a wave propagating in the direction specified by the propagation vector

$$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z \tag{2.77}$$

because the scalar product of $\mathbf{k}$ with the position vector

$$\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$$

equals $k_x x + k_y y + k_z z$ and is k_0 times the perpendicular distance from the origin to a plane normal to the vector $\mathbf{k}$, as illustrated in Fig. 2.11. The $\mathbf{k}$ vector may also be written as $\mathbf{k} = \mathbf{n}k_0$, where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{k}$ and k_0 is the magnitude of $\mathbf{k}$ by virtue of (2.75).

Although (2.76) gives a possible solution for E_x, this is not the complete solution for the electric field. Similar solutions for E_y and E_z may be found. The three components of $\mathbf{E}$ are not independent since the divergence relation $\nabla \cdot \mathbf{E} = 0$ must hold in free space. This constraint means

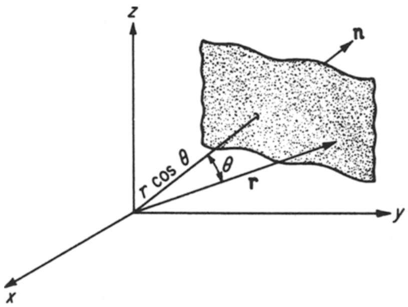

FIGURE 2.11
Illustration of plane normal to vector **k**.

that only two components of **E** can have arbitrary amplitudes. However, for $\nabla \cdot \mathbf{E}$ to vanish everywhere, all components of **E** must have the same spatial dependence, and hence appropriate solutions for E_y and E_z are

$$E_y = Be^{-j\mathbf{k}\cdot\mathbf{r}} \qquad E_z = Ce^{-j\mathbf{k}\cdot\mathbf{r}}$$

with B and C amplitude coefficients. Let $\mathbf{E}_0$ be the vector $\mathbf{a}_x A + \mathbf{a}_y B + \mathbf{a}_z C$; then the total solution for **E** may be written in vector form as

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{2.78}$$

The divergence condition gives

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_0 \cdot \nabla e^{-j\mathbf{k}\cdot\mathbf{r}} = -j\mathbf{k} \cdot \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = 0$$

or

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \tag{2.79}$$

since $\nabla e^{-j\mathbf{k}\cdot\mathbf{r}} = -j\mathbf{k}e^{-j\mathbf{k}\cdot\mathbf{r}}$, as may be verified by expansion in rectangular coordinates. The divergence condition is seen to constrain the amplitudes A, B, C so that the vector $\mathbf{E}_0$ is perpendicular to the direction of propagation as specified by **k**. The solution (2.78) is called a uniform plane wave since the constant-phase surfaces given by $\mathbf{k} \cdot \mathbf{r} = \text{const}$ are planes and the field **E** does not vary on a constant-phase plane.

The solution for **H** is obtained from Maxwell's equation

$$\nabla \times \mathbf{E} = -j\omega\mu_0 \mathbf{H}$$

which gives

$$\begin{aligned}\mathbf{H} &= -\frac{1}{j\omega\mu_0}\nabla \times \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \frac{1}{j\omega\mu_0}\mathbf{E}_0 \times \nabla e^{-j\mathbf{k}\cdot\mathbf{r}} \\ &= \frac{1}{\omega\mu_0}\mathbf{k} \times \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \frac{k_0}{\omega\mu_0}\mathbf{n} \times \mathbf{E} \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}\,\mathbf{n} \times \mathbf{E} = Y_0 \mathbf{n} \times \mathbf{E} \end{aligned} \tag{2.80}$$

where $Y_0 = \sqrt{\epsilon_0/\mu_0}$ has the dimensions of an admittance and is called the intrinsic admittance of free space. The reciprocal $Z_0 = 1/Y_0$ is called the

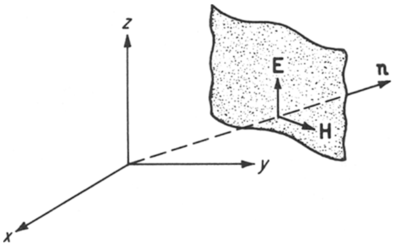

FIGURE 2.12
Space relationship between **E**, **H**, and **n** in a TEM wave.

intrinsic impedance of free space. Note that **H** is perpendicular to **E** and to **n**, and hence both **E** and **H** lie in the constant-phase planes. For this reason this type of wave is called a transverse electromagnetic wave (TEM wave). The spatial relationship between **E**, **H**, and **n** is illustrated in Fig. 2.12.

The physical electric field corresponding to the phasor representation (2.78) is

$$\mathbf{E} = \mathrm{Re}\left(\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}+j\omega t}\right) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) \qquad (2.81)$$

where, for simplicity, $\mathbf{E}_0$ has been assumed to be real. The wavelength is the distance the wave must propagate to undergo a phase change of 2π. If we let λ_0 denote the wavelength in free space, it follows that

$$|\mathbf{k}|\lambda_0 = k_0\lambda_0 = 2\pi$$

so that

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0} \qquad (2.82)$$

This result is the familiar relationship between wavelength λ_0, frequency $f = \omega/2\pi$, and velocity c in free space. A wavelength in a direction other than that along the direction of propagation **n** may also be defined. For example, along the direction of the x axis the wavelength is

$$\lambda_x = \frac{2\pi}{k_x} \qquad (2.83)$$

and since k_x is less than k_0, λ_x is greater than λ_0. The phase velocity is the velocity with which an observer would have to move in order to see a constant phase. From (2.81) it is seen that the phase of **E** is constant as long as $\mathbf{k}\cdot\mathbf{r} - \omega t$ is constant. If the angle between **k** and **r** is θ, then $\mathbf{k}\cdot\mathbf{r} - \omega t = k_0 r\cos\theta - \omega t$. Differentiating the relation

$$k_0 r\cos\theta - \omega t = \text{const}$$

gives

$$\frac{dr}{dt} = v_p = \frac{\omega}{k_0\cos\theta} \qquad (2.84)$$

for the phase velocity v_p in the direction **r**. Along the direction of propagation, $\cos\theta = 1$ and $v_p = \omega/k_0 = c$. In other directions, the phase velocity is

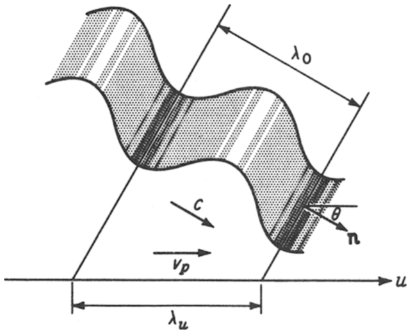

FIGURE 2.13
A wave propagating obliquely to the u axis.

greater than c. These results may be understood by reference to Fig. 2.13. When the wave has moved a distance λ_0 along the direction $\mathbf{n}$, the constant-phase-plane intersection with the u axis has moved a distance $\lambda_u = \lambda_0 \sec\theta$ along the direction u. For this reason the wavelength and phase velocity along u are greater by a factor $\sec\theta$ than the corresponding quantities measured along the direction of propagation $\mathbf{n}$.

The time-average rate of energy flow per unit area in the direction $\mathbf{n}$ is given by

$$P = \tfrac{1}{2}\,\mathrm{Re}\,\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n} = \tfrac{1}{2}\,\mathrm{Re}\,Y_0\mathbf{E}\times(\mathbf{n}\times\mathbf{E}^*)\cdot\mathbf{n} = \tfrac{1}{2}Y_0\mathbf{E}_0\cdot\mathbf{E}_0^* \tag{2.85}$$

The time-average energy densities in the electric and magnetic fields of a TEM wave are, respectively,

$$U_e = \frac{\epsilon_0}{4}\mathbf{E}\cdot\mathbf{E}^* = \frac{\epsilon_0}{4}\mathbf{E}_0\cdot\mathbf{E}_0^*$$

$$U_m = \frac{\mu_0}{4}\mathbf{H}\cdot\mathbf{H}^* = \frac{\mu_0}{4}Y_0^2(\mathbf{n}\times\mathbf{E})\cdot(\mathbf{n}\times\mathbf{E}^*) = \frac{\epsilon_0}{4}\mathbf{E}_0\cdot\mathbf{E}_0^* = U_e$$

and are seen to be equal. Since power is a flow of energy, the velocity v_g of energy propagation is such that

$$(U_e + U_m)v_g = P$$

or

$$v_g = \frac{P}{U_e + U_m} = \frac{\frac{1}{2}Y_0\mathbf{E}_0\cdot\mathbf{E}_0^*}{\frac{1}{2}\epsilon_0\mathbf{E}_0\cdot\mathbf{E}_0^*} = \frac{Y_0}{\epsilon_0} = c \tag{2.86}$$

Thus, for a TEM wave in free space, the energy in the field is transported with a velocity $c = 3\times10^8$ m/s, which is also the phase velocity. Since the phase velocity is independent of frequency, a modulated carrier or signal will have all its frequency components propagated with the same velocity c. Hence the signal velocity is also the velocity of light c. Later on, in the study of waveguides, situations arise where the phase velocity is dependent on frequency and consequently is not equal to the velocity of energy propagation or the signal velocity.

2.8 REFLECTION FROM A DIELECTRIC INTERFACE

In Fig. 2.14 the half-space $z \geq 0$ is filled with a dielectric medium with permittivity ϵ (dielectric constant $\epsilon_r = \epsilon/\epsilon_0$; index of refraction $\eta = \sqrt{\epsilon_r}$). A TEM wave is assumed incident from the region $z < 0$. Without loss in generality, the xy axis may be oriented so that the unit vector $\mathbf{n}_1$ specifying the direction of incidence lies in the xz plane. It is convenient to solve this problem as two special cases, namely (1) parallel polarization, where the electric field of the incident wave is coplanar with $\mathbf{n}_1$ and the interface normal, i.e., lies in the xz plane, and (2) perpendicular polarization, where the electric field of the incident wave is perpendicular to the plane of incidence as defined by $\mathbf{n}_1$ and the interface normal, i.e., along the y axis. An incident TEM wave with arbitrary polarization can always be decomposed into a linear sum of perpendicular and parallel polarized waves. The reason for treating the two polarizations separately is that the reflection and transmission coefficients, to be defined, are different for the two cases.

1 Parallel Polarization

Let the incident TEM wave be

$$\mathbf{E}_i = \mathbf{E}_1 e^{-jk_0\mathbf{n}_1\cdot\mathbf{r}} \qquad \mathbf{H}_i = Y_0\mathbf{n}_1 \times \mathbf{E}_i \tag{2.87}$$

where $\mathbf{E}_1$ lies in the xz plane. Part of the incident power will be reflected, and the remainder will be transmitted into the dielectric medium. Let the reflected TEM wave be

$$\mathbf{E}_r = \mathbf{E}_2 e^{-jk_0\mathbf{n}_2\cdot\mathbf{r}} \qquad \mathbf{H}_r = Y_0\mathbf{n}_2 \times \mathbf{E}_r \tag{2.88}$$

where $\mathbf{n}_2$ and $\mathbf{E}_2$ are to be determined. In the dielectric medium the solution for a TEM wave is the same as that in free space, but with ϵ_0 replaced by ϵ. Thus, in place of $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and $Y_0 = \sqrt{\epsilon_0/\mu_0}$, the parameters $k = \omega\sqrt{\mu_0\epsilon} = \eta k_0$ and $Y = \sqrt{\epsilon/\mu_0} = \eta Y_0$ are used, where $\eta = \sqrt{\epsilon_r}$ is the index of refraction. The transmitted wave in the dielectric may be

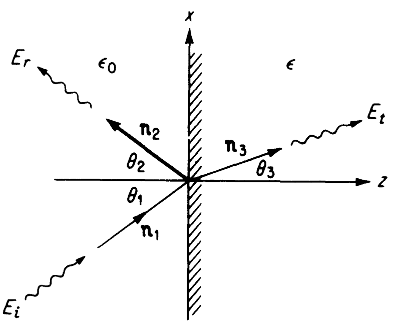

FIGURE 2.14
Plane wave incident on a dielectric interface.

expressed by

$$\mathbf{E}_t = \mathbf{E}_3 e^{-jk\mathbf{n}_3\cdot\mathbf{r}} \qquad \mathbf{H}_t = Y\mathbf{n}_3 \times \mathbf{E}_t \tag{2.89}$$

with $\mathbf{E}_3$ and $\mathbf{n}_3$ as yet unknown.

The boundary conditions to be applied are the continuity of the tangential components of the electric and magnetic fields at the interface plane $z = 0$. These components must be continuous for all values of x and y on the $z = 0$ plane, and this is possible only if the fields on adjacent sides of the boundary have the same variation with x and y. Hence we must have

$$k_0 n_{1x} = k_0 n_{2x} = k n_{3x} = \eta k_0 n_{3x} \tag{2.90}$$

i.e., the propagation phase constant along x must be the same for all waves. Since n_{1y} was chosen as zero, it follows that $n_{2y} = n_{3y} = 0$ also. The unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ may be expressed as

$$\mathbf{n}_1 = \mathbf{a}_x \sin\theta_1 + \mathbf{a}_z \cos\theta_1$$

$$\mathbf{n}_2 = \mathbf{a}_x \sin\theta_2 + \mathbf{a}_z \cos\theta_2$$

$$\mathbf{n}_3 = \mathbf{a}_x \sin\theta_3 + \mathbf{a}_z \cos\theta_3$$

Equation (2.90) gives

$$\sin\theta_1 = \sin\theta_2$$

or

$$\theta_1 = \theta_2 \tag{2.91}$$

which is the well-known Snell's law of reflection; in addition, (2.90) gives

$$\sin\theta_1 = \eta \sin\theta_3 \tag{2.92}$$

which is also a well-known result specifying the angle of refraction θ_3 in terms of the angle of incidence θ_1 and the index of refraction η.

The incident electric field $\mathbf{E}_1$ has components $\mathbf{E}_{1x} = E_1 \cos\theta_1$,

$$E_{1z} = -E_1 \sin\theta_1$$

since $\mathbf{n}_1 \cdot \mathbf{E}_1$ must equal zero. Note that E_1 is used to denote the magnitude of the vector $\mathbf{E}_1$. Since the incident electric field has no y component, the reflected and transmitted electric fields also have zero y components.† Expressing all fields in component form, i.e.,

$$E_{2x} = E_2 \cos\theta_2$$

$E_{2z} = E_2 \sin\theta_2$, $E_{3x} = E_3 \cos\theta_3$, $E_{3z} = -E_3 \sin\theta_3$, and imposing the boundary condition of continuity of the x component at $z = 0$ yields the relation

$$E_1 \cos\theta_1 + E_2 \cos\theta_2 = E_3 \cos\theta_3$$

†If the reflected and transmitted electric fields were assumed to have a y component, the boundary conditions which must apply would show that these are, indeed, zero.

or
$$(E_1 + E_2)\cos\theta_1 = E_3\sqrt{1 - \sin^2\theta_3} = E_3\frac{\sqrt{\epsilon_r - \sin^2\theta_1}}{\eta} \tag{2.93}$$

by using (2.91) and (2.92). Apart from the propagation factor, the magnetic field is given by

$$\mathbf{H}_1 = Y_0\mathbf{n}_1 \times \mathbf{E}_1 = Y_0\mathbf{a}_y(-n_{1x}E_{1z} + n_{1z}E_{1x}) = Y_0\mathbf{a}_y E_1$$
$$\mathbf{H}_2 = -Y_0\mathbf{a}_y E_2$$
$$\mathbf{H}_3 = Y\mathbf{a}_y E_3$$

and has only a y component. Continuity of this magnetic field at the boundary requires that

$$Y_0(E_1 - E_2) = YE_3 = \eta Y_0 E_3 \tag{2.94}$$

If a reflection coefficient Γ_1 and a transmission coefficient T_1 are introduced according to the following relations:

$$\Gamma_1 = \frac{\text{amplitude of reflected electric field}}{\text{amplitude of incident electric field}} = \frac{E_2}{E_1} \tag{2.95a}$$

$$T_1 = \frac{\text{amplitude of transmitted electric field}}{\text{amplitude of incident electric field}} = \frac{E_3}{E_1} \tag{2.95b}$$

then the boundary conditions (2.93) and (2.94) may be expressed as

$$1 + \Gamma_1 = T_1\frac{(\epsilon_r - \sin^2\theta_1)^{1/2}}{\eta\cos\theta_1} \tag{2.96a}$$

$$1 - \Gamma_1 = \eta T_1 \tag{2.96b}$$

These equations may be solved to give the Fresnel reflection and transmission coefficients for the case of parallel polarization, namely,

$$\Gamma_1 = \frac{(\epsilon_r - \sin^2\theta_1)^{1/2} - \epsilon_r\cos\theta_1}{(\epsilon_r - \sin^2\theta_1)^{1/2} + \epsilon_r\cos\theta_1} \tag{2.97a}$$

$$T_1 = \frac{2\eta\cos\theta_1}{(\epsilon_r - \sin^2\theta_1)^{1/2} + \epsilon_r\cos\theta_1} \tag{2.97b}$$

An interesting feature of Γ_1 is that it vanishes for an angle of incidence $\theta_1 = \theta_b$, called the Brewster angle, where, from (2.97a),

$$\epsilon_r - \sin^2\theta_b = \epsilon_r^2\cos^2\theta_b$$

or
$$\sin\theta_b = \left(\frac{\epsilon_r}{\epsilon_r + 1}\right)^{1/2} \tag{2.98}$$

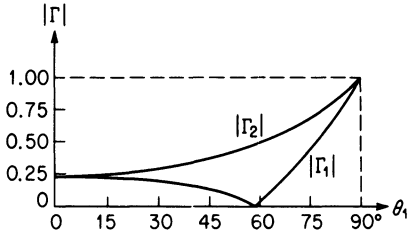

FIGURE 2.15
Modulus of reflection coefficient at a dielectric interface for $\epsilon_r = 2.56$, $|\Gamma_1|$ parallel polarization, $|\Gamma_2|$ perpendicular polarization.

At this particular angle of incidence all the incident power is transmitted into the dielectric medium. In Fig. 2.15 the reflection coefficient Γ_1 is plotted as a function of θ_1 for polystyrene, for which $\epsilon_r = 2.56$.

2 Perpendicular Polarization

For perpendicular polarization the roles of electric and magnetic fields are interchanged so that the electric field has only a y component. The fields may, however, still be expressed in the form given by (2.87) to (2.89), but with $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ having y components only. As in the previous case, the boundary conditions must hold for all values of x and y on the $z = 0$ plane. Therefore Snell's laws of reflection and refraction again result; i.e., (2.91) and (2.92) must be satisfied. In place of the boundary conditions (2.93) and (2.94), we have

$$E_1 + E_2 = E_3 \tag{2.99a}$$

$$Y_0(E_1 - E_2)\cos\theta_1 = YE_3\cos\theta_3 \tag{2.99b}$$

Introducing the following reflection and transmission coefficients:

$$\Gamma_2 = \frac{E_2}{E_1} \qquad T_2 = \frac{E_3}{E_1}$$

into (2.99) yields

$$1 + \Gamma_2 = T_2 \tag{2.100a}$$

$$1 - \Gamma_2 = T_2\frac{\left(\epsilon_r - \sin^2\theta_1\right)^{1/2}}{\cos\theta_1} \tag{2.100b}$$

The Fresnel reflection and transmission coefficients for the case of perpendicular polarization thus are

$$\Gamma_2 = \frac{\cos\theta_1 - \left(\epsilon_r - \sin^2\theta_1\right)^{1/2}}{\left(\epsilon_r - \sin^2\theta_1\right)^{1/2} + \cos\theta_1} \tag{2.101a}$$

$$T_2 = \frac{2\cos\theta_1}{\left(\epsilon_r - \sin^2\theta_1\right)^{1/2} + \cos\theta_1} \tag{2.101b}$$

A notable difference for this case is the nonexistence of a Brewster angle for which Γ_2 vanishes. For comparison with the case of parallel polarization, Γ_2 is plotted in Fig. 2.15 for $\epsilon_r = 2.56$.

2.9 REFLECTION FROM A CONDUCTING PLANE

The essential features of the behavior of the electromagnetic field at the surface of a good conductor may be deduced from an analysis of the simple problem of a TEM wave incident normally onto a conducting plane. The problem is illustrated in Fig. 2.16, which shows a medium with parameters ϵ, μ, σ filling the half-space $z \ge 0$. Let the electric field be polarized along the x axis so that the incident and reflected fields may be expressed as

$$\begin{aligned} \mathbf{E}_i &= E_1\mathbf{a}_x e^{-jk_0z} \\ \mathbf{H}_i &= Y_0E_1\mathbf{a}_y e^{-jk_0z} \\ \mathbf{E}_r &= \Gamma E_1\mathbf{a}_x e^{+jk_0z} \\ \mathbf{H}_r &= -Y_0\Gamma E_1\mathbf{a}_y e^{+jk_0z} \end{aligned} \tag{2.102b}$$

where Γ is the reflection coefficient.

In the conducting medium the conduction current $\sigma\mathbf{E}$ is much greater than the displacement current $j\omega\epsilon\mathbf{E}$, so that Helmholtz's equation reduces to (2.50); i.e.,

$$\nabla^2\mathbf{E} - j\omega\mu\sigma\mathbf{E} = 0$$

The transmitted field is a solution of

$$\left(\frac{\partial^2}{\partial z^2} - j\omega\mu\sigma\right)\mathbf{E}_t = 0$$

since no x or y variation is assumed. The solution for a wave with an x component only and propagating in the z direction is

$$\mathbf{E}_t = E_3\mathbf{a}_x e^{-\gamma z} \tag{2.103a}$$

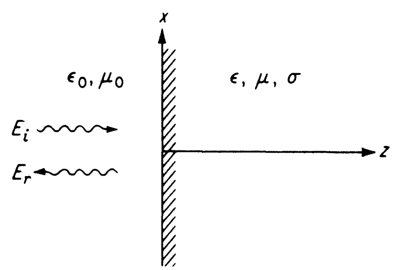

FIGURE 2.16
A TEM wave incident normally on a conducting plane.

with a corresponding magnetic field

$$\mathbf{H}_t = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E}_t = \frac{\gamma}{j\omega\mu}\mathbf{a}_y E_3 e^{-\gamma z} \tag{2.103b}$$

where

$$\gamma = (j\omega\mu\sigma)^{1/2} = \frac{1+j}{\delta_s} \tag{2.104}$$

and the skin depth $\delta_s = (\omega\mu\sigma/2)^{-1/2}$. The propagation constant $\gamma = \alpha + j\beta$ has equal phase and attenuation constants. In the conductor the fields decay by an amount e^{-1} in a distance of one skin depth δ_s, which is a very small distance for metals at microwave frequencies (about 10^{-5} cm). The intrinsic impedance of the metal is Z_m, where

$$Z_m = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{(j\omega\mu\sigma)^{1/2}} = \frac{1+j}{\sigma\delta_s} \tag{2.105}$$

and is very small compared with the intrinsic impedance $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ of free space. For example, for copper at 10^4 MHz, $Z_m = 0.026(1+j)\ \Omega$ as compared with 377 Ω for Z_0. Note that (2.103*b*) may be written as

$$\mathbf{H}_t = \frac{1}{Z_m}\mathbf{a}_y E_3 e^{-\gamma z} = Y_m \mathbf{a}_y E_3 e^{-\gamma z}$$

which shows that the ratio of the magnitudes of the electric field to the magnetic field for a TEM wave in a conductor is the intrinsic impedance Z_m.

Returning to the boundary-value problem and imposing the boundary conditions of continuity of tangential fields at the boundary plane $z = 0$ give

$$(1+\Gamma)E_1 = E_3 = TE_1 \tag{2.106a}$$

$$(1-\Gamma)Y_0 E_1 = H_3 = Y_m E_3 = Y_m T E_1 \tag{2.106b}$$

where $E_3/E_1 = T$, the transmission coefficient. Solving (2.106) for the reflection coefficient Γ and T yields

$$\Gamma = \frac{Z_m - Z_0}{Z_m + Z_0} \tag{2.107a}$$

$$T = 1 + \Gamma = \frac{2Z_m}{Z_m + Z_0} \tag{2.107b}$$

Since $|Z_m|$ is small compared with Z_0, the reflection coefficient Γ is almost equal to -1 and the transmission coefficient T is very small. Almost all the incident power is reflected from the metallic boundary. As the conductivity σ is made to approach infinity, the impedance Z_m approaches zero and in the limit $\Gamma = -1$ and $T = 0$. Hence, for a perfect conductor, the tangential electric field at the surface is zero and the tangential magnetic field has a value equal to twice that of the incident wave.

The current density in the conductor is $\mathbf{J} = \sigma \mathbf{E}_t = \sigma T E_1 \mathbf{a}_x e^{-\gamma z}$. The total current per unit width of conductor along y is

$$\mathbf{J}_s = \int_0^\infty \mathbf{J}\, dz = \sigma T E_1 \mathbf{a}_x \int_0^\infty e^{-\gamma z}\, dz = \frac{\sigma T E_1 \mathbf{a}_x}{\gamma} \text{ A/m}$$

This result may also be expressed in the following form:

$$\mathbf{J}_s = \frac{2\sigma Z_m^2 E_1}{(Z_m + Z_0) j\omega\mu} \mathbf{a}_x \tag{2.108}$$

by substituting for T from (2.107*b*) and replacing γ by $j\omega\mu/Z_m$ from (2.105). As $\sigma \to \infty$, the limiting value of $\mathbf{J}_s$ becomes

$$\mathbf{J}_s = \frac{2E_1}{Z_0} \mathbf{a}_x = 2Y_0 E_1 \mathbf{a}_x \tag{2.109}$$

since $Z_m \to 0$ and $\sigma Z_m^2 \to j\omega\mu$. This current exists only on the surface of the conductor since, as $\sigma \to \infty$, the skin depth $\delta_s \to 0$; that is, the field decays infinitely fast with distance into the conductor. When σ is infinite, $\Gamma = -1$ and the total tangential magnetic field at the surface is $2Y_0 E_1 \mathbf{a}_y$ and equal in magnitude to $\mathbf{J}_s$. In vector form the boundary conditions at the surface of a perfect conductor are thus seen to be

$$\mathbf{n} \times \mathbf{E} = 0 \tag{2.110a}$$

$$\mathbf{n} \times \mathbf{H} = \mathbf{J}_s \tag{2.110b}$$

where $\mathbf{n}$ is a unit outward normal at the conductor surface.

For finite conductivity the current density at the surface is $\sigma T E_1$ and the magnetic field at the surface is $Y_m T E_1$. In terms of these quantities the total current per unit width may be expressed as

$$\mathbf{J}_s = \frac{\sigma T E_1}{\gamma} \mathbf{a}_x = \frac{\sigma Z_m}{\gamma} (Y_m T E_1) \mathbf{a}_x = Y_m T E_1 \mathbf{a}_x$$

In other words, the total current per unit width is equal to the tangential magnetic field at the surface.

The time-average power transmitted into the conductor per unit area is given by the real part of one-half of the complex Poynting vector at the surface, and is

$$P_t = \tfrac{1}{2} \operatorname{Re} \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{a}_z = \tfrac{1}{2} T T^* E_1 E_1^* \operatorname{Re} Y_m = \tfrac{1}{4} T T^* E_1 E_1^* \sigma \delta_s \tag{2.111}$$

The reader may readily verify that this is equal to the result obtained from a volume integral of $\mathbf{J} \cdot \mathbf{J}^*$; that is,

$$P_t = \frac{1}{2\sigma} \int_0^\infty \mathbf{J} \cdot \mathbf{J}^*\, dz$$

Equation (2.111) may be simplified with little error by making the following approximation:

$$\sigma TT^* = \frac{4\sigma Z_m Z_m^*}{(Z_m + Z_0)(Z_m + Z_0)^*}$$

$$\approx \frac{4\sigma Z_m Z_m^*}{Z_0^2} = \frac{8}{\sigma \delta_s^2 Z_0^2}$$

whence (2.111) becomes

$$P_t \approx \frac{1}{2} \frac{(2Y_0 E_1)(2Y_0 E_1^*)}{\sigma \delta_s} \tag{2.112}$$

Note that $2Y_0 E_1$ is the value of the magnetic field, tangent to the surface, that would exist if σ were infinite. Hence an excellent approximate technique for evaluating power loss in a conductor is to find the tangential magnetic field, say H_t, that would exist for a perfect conductor, and then compute the power loss according to the relation

$$P_t = \tfrac{1}{2}\,\mathrm{Re}(H_t H_t^* Z_m) = \tfrac{1}{2}\,\mathrm{Re}(J_s J_s^* Z_m) \tag{2.113}$$

This procedure is equivalent to assuming that the metal exhibits a surface impedance Z_m and the current is essentially the same as that which would exist for infinite conductivity.

The procedure outlined above for power-loss calculations is widely used in microwave work. Although the derivation was based on a consideration of a very special boundary-value problem, the same conclusions result for more complex structures such as conducting spheres and cylinders. In general, the technique of characterizing the metal by a surface impedance Z_m and assuming that the current $\mathbf{J}_s$ is the same as that for infinite conductivity is valid as long as the conductor surface has a radius of curvature at least a few skin depths in magnitude.

2.10 POTENTIAL THEORY

The wave solutions presented in the previous sections have all been source-free solutions; i.e., the nature of the sources giving rise to the field was not considered. When it is necessary to consider the specific field generated by a given source, as in antenna problems, waveguide and cavity coupling, etc., this is greatly facilitated by introducing an auxiliary vector potential function **A**. As will be seen, the vector potential **A** is determined by the current source, and the total electromagnetic field may be derived from **A**.

Since $\nabla \cdot \mathbf{B} = 0$ always, this condition will hold identically if **B** is expressed as the curl of a vector potential **A** since the divergence of the curl of a vector is identically zero. Thus let

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.114}$$

The assumed time dependence $e^{j\omega t}$ is not written out explicitly in (2.114) since this is a phasor representation. The curl equation for $\mathbf{E}$ gives

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\nabla \times \mathbf{A}$$

or

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0$$

Now the curl of the gradient of a scalar function Φ is identically zero; so the general integral of the above equation is

$$\mathbf{E} + j\omega\mathbf{A} = -\nabla\Phi$$

or

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi \tag{2.115}$$

Substituting this expression into the $\nabla \times \mathbf{H}$ equation gives

$$\nabla \times \mathbf{H} = \frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = j\omega\epsilon\mathbf{E} + \mathbf{J} = \omega^2\epsilon\mathbf{A} - j\omega\epsilon\nabla\Phi + \mathbf{J} \tag{2.116}$$

Up to this point the divergences of $\mathbf{A}$ and $\nabla\Phi$ have not been specified [note that (2.114) specifies the curl of $\mathbf{A}$ only]. Therefore a relation between $\nabla \cdot \mathbf{A}$ and Φ may be chosen so as to simplify (2.116). Expanding $\nabla \times \nabla \times \mathbf{A}$ to give $\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A}$ enables us to write (2.116) as

$$\nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} = k^2\mathbf{A} - j\omega\epsilon\mu\nabla\Phi + \mu\mathbf{J}$$

where $k^2 = \omega^2\mu\epsilon$. If now the following condition is specified:

$$\nabla\nabla \cdot \mathbf{A} = -j\omega\epsilon\mu\nabla\Phi$$

or

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon\Phi \tag{2.117}$$

this equation simplifies to

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \tag{2.118}$$

Thus $\mathbf{A}$ is a solution of the inhomogeneous Helmholtz equation, the current $\mathbf{J}$ being the source term. The condition imposed on $\nabla \cdot \mathbf{A}$ and Φ in (2.117) is called the Lorentz condition in honor of the man first to propose its use.

In the preceding derivation three of Maxwell's equations have been used and are therefore satisfied. The fourth equation, $\nabla \cdot \mathbf{D} = \rho$, must also hold, and this will be shown to be the case provided the Lorentz condition is obeyed. Hence the three equations (2.114), (2.115), and (2.118), together with the Lorentz condition (2.117), are fully equivalent to Maxwell's equations. To verify the equation $\nabla \cdot \mathbf{D} = \rho$, take the divergence of (2.115) to obtain

$$\nabla \cdot \epsilon\mathbf{E} = -j\omega\epsilon\nabla \cdot \mathbf{A} - \epsilon\nabla^2\Phi \tag{2.119}$$

where ϵ is a constant. Using the Lorentz condition yields

$$\nabla \cdot \mathbf{D} = -j\omega\epsilon\nabla \cdot \mathbf{A} - \nabla^2\frac{\nabla \cdot \mathbf{A}}{-j\omega\mu} = \frac{1}{j\omega\mu}\nabla \cdot (\nabla^2\mathbf{A} + k^2\mathbf{A}) = -\frac{1}{j\omega}\nabla \cdot \mathbf{J}$$

by using (2.118) and noting that $\nabla^2\nabla \cdot \mathbf{A} = \nabla \cdot \nabla^2\mathbf{A}$; that is, these differential operators commute. Now $\nabla \cdot \mathbf{J} = -j\omega\rho$ from the continuity equation;

so we obtain

$$\nabla \cdot \mathbf{D} = -\frac{1}{j\omega}(-j\omega\rho) = \rho$$

If, instead of eliminating Φ in (2.119), $\nabla \cdot \mathbf{A}$ is eliminated by use of the Lorentz condition, we get

$$\nabla \cdot \mathbf{D} = \rho = -j\omega\epsilon(-j\omega\mu\epsilon\Phi) - \epsilon\nabla^2\Phi$$

or

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho}{\epsilon} \tag{2.120}$$

Hence the scalar potential Φ is a solution of the inhomogeneous scalar Helmholtz equation, with the charge density ρ as a source term.

For the time-varying field, $\mathbf{J}$ and ρ are not independent, and hence the field can be determined in terms of $\mathbf{A}$ and $\mathbf{J}$ alone. The scalar potential can always be found from the Lorentz relation, and ρ from the continuity equation, but explicit knowledge of these is not required in order to solve radiation problems. For convenience, the pertinent equations are summarized here:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{2.121a}$$

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi = -j\omega\mathbf{A} + \frac{\nabla\nabla \cdot \mathbf{A}}{j\omega\mu\epsilon} = \frac{k^2\mathbf{A} + \nabla\nabla \cdot \mathbf{A}}{j\omega\mu\epsilon} \tag{2.121b}$$

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \tag{2.121c}$$

where the Lorentz condition was used to eliminate $\nabla\Phi$ in (2.121*b*). Note that, in rectangular coordinates, (2.121*c*) is three scalar equations of the form

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

but that, in other coordinate systems, $\nabla^2\mathbf{A}$ must be expanded according to the relation $\nabla^2\mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A}$.

The simplest solution to (2.121*c*) is that for an infinitesimal current element $\mathbf{J}(x', y', z') = \mathbf{J}(\mathbf{r}')$ located at the point x', y', z', as specified by the position vector $\mathbf{r}' = \mathbf{a}_x x' + \mathbf{a}_y y' + \mathbf{a}_z z'$, as in Fig. 2.17. This solution is

$$\mathbf{A}(x, y, z) = \mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\mathbf{J}(\mathbf{r}')\frac{e^{-jkR}}{R}\,dV' \tag{2.122}$$

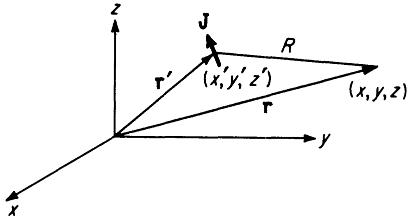

FIGURE 2.17
Coordinates used to describe vector potential from a current sheet.

where $R = |\mathbf{r} - \mathbf{r}'|$ is the magnitude of the distance from the source point to the field point at which $\mathbf{A}$ is evaluated; i.e.,

$$R = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{1/2}$$

and $\mathbf{J}(\mathbf{r}')\,dV'$ is the total source strength. In terms of this fundamental solution, the vector potential from a general current distribution may be obtained by superposition. Thus, adding up all the contributions from each infinitesimal current element gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(x', y', z')\frac{e^{-jkR}}{R}\,dx'\,dy'\,dz' = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}\,dV' \tag{2.123}$$

where the integration is over the total volume occupied by the current. Note that the solution for $\mathbf{A}$ as given by (2.122) is a spherical wave propagating radially outward from $\mathbf{J}$ and with an amplitude falling off as $1/R$. The solution (2.123) is a superposition of such elementary spherical waves.

★2.11 DERIVATION OF SOLUTION FOR VECTOR POTENTIAL

In this section a detailed derivation of the solution to the inhomogeneous Helmholtz equation for a unit current source is given. A unit source is a source of unit strength, localized at a point in space (a familiar example is a point charge). Such a unit source in a three-dimensional space is a generalization of a unit current impulse localized at a time t' along the time coordinate. A current pulse is represented by the Dirac delta function $\delta(t - t')$ in circuit theory, where $\delta(t - t')$ has the property

$$\delta(t - t') = 0 \qquad t \neq t' \tag{2.124a}$$

and at $t = t'$ it becomes infinite but is integrable to give

$$\int_{t'-\tau}^{t'+\tau} \delta(t - t')\,dt = 1 \tag{2.124b}$$

A further property is that, for any function $f(t)$ which is continuous at t',

$$\int_{t'-\tau}^{t'+\tau} f(t)\delta(t - t')\,dt = f(t') \tag{2.124c}$$

This result follows since τ can be chosen so small that, in the interval $t' - \tau < t < t' + \tau$, the function $f(t)$ differs by a vanishing amount from $f(t')$ since $f(t)$ is continuous at t'. Hence (2.124c) may be written as

$$f(t')\int_{t'-\tau}^{t'+\tau} \delta(t - t')\,dt = f(t')$$

by virtue of (2.124b).

As the preceding discussion has shown, the delta function is a convenient mathematical way to represent a source of unit strength localized at a point along a coordinate axis, in the above example along the time axis. In an N-dimensional space a product of N delta functions, one for each coordinate, may be used to represent a unit source. Thus, in three dimensions, a unit source is represented by

$$\delta(x - x')\delta(y - y')\delta(z - z') = \delta(\mathbf{r} - \mathbf{r}') \tag{2.125}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is an abbreviated notation for the product of the three one-dimensional delta functions. The source function $\delta(\mathbf{r} - \mathbf{r}')$ has the following properties:

$$\delta(\mathbf{r} - \mathbf{r}') = 0 \qquad \mathbf{r} \neq \mathbf{r}' \tag{2.126a}$$

$$\int_V \delta(\mathbf{r} - \mathbf{r}')\, dV = \begin{cases} 1 & \mathbf{r}' \text{ in } V \\ 0 & \mathbf{r}' \text{ not in } V \end{cases} \tag{2.126b}$$

$$\int_V \mathbf{F}(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')\, dV = \begin{cases} \mathbf{F}(\mathbf{r}') & \mathbf{r}' \text{ in } V \\ 0 & \mathbf{r}' \text{ not in } V \end{cases} \tag{2.126c}$$

where $\mathbf{F}$ is an arbitrary vector (or scalar) function that is continuous at $\mathbf{r}'$, that is, at x', y', z'. These properties follow from the properties of the one-dimensional delta functions that make up $\delta(\mathbf{r} - \mathbf{r}')$.

A unit current source directed along the unit vector $\mathbf{a}$ at $\mathbf{r}'$ may be expressed as $\mathbf{J} = \mathbf{a}\delta(\mathbf{r} - \mathbf{r}')$. The vector potential is a solution of

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{a}\delta(\mathbf{r} - \mathbf{r}') \tag{2.127}$$

Since the current is in the direction $\mathbf{a}$, the vector potential must also be in this direction, and hence $\mathbf{A} = A\mathbf{a}$. Equation (2.127) may therefore be written as a scalar equation:

$$\nabla^2 A + k^2 A = -\mu\delta(\mathbf{r} - \mathbf{r}') \tag{2.128}$$

At all points $\mathbf{r} \neq \mathbf{r}'$, A is a solution of

$$\nabla^2 A + k^2 A = 0 \tag{2.129}$$

If the source point $\mathbf{r}'$ is considered as the origin in a spherical coordinate system, then, since no angle variables occur in the source term in (2.128), the solution for A must have spherical symmetry about the source point $\mathbf{r}'$. Thus, in terms of the spherical radial coordinate $R = |\mathbf{r} - \mathbf{r}'|$, which is the radial distance from the origin at $\mathbf{r}'$, (2.129) is a function of R only and may be written as

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial A}{\partial R}\right) + k^2 A = 0$$

or

$$\frac{d^2A}{dR^2} + \frac{2}{R}\frac{dA}{dR} + k^2 A = 0 \tag{2.130}$$

after expressing the R-dependent part of ∇^2 in spherical coordinates. In anticipation of a spherical-wave solution, let $A = f(R)e^{-jkR}$. Substitution in (2.130) leads to the following equation for $f(R)$:

$$\frac{d^2f}{dR} + \left(\frac{2}{R} - 2jk\right)\frac{df}{dR} - \frac{2jk}{R}f = 0$$

which is readily verified to have the solution $f = C/R$, where C is an arbitrary constant. Consequently, the solution to (2.129) is $A = Ce^{-jkR}/R$. This solution is singular at $R = 0$, and the singularity must correspond to that of the source term at this point.

To determine the constant C, integrate (2.128) throughout a small sphere of radius r_0 centered on $\mathbf{r}'$ and use the delta-function property (2.126*b*) to obtain

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r_0}(\nabla^2 A + k^2 A)R^2 \sin\theta \, d\theta \, d\phi \, dR$$

$$= \int_V (\nabla^2 A + k^2 A)\, dV = -\mu \int_V \delta(\mathbf{r} - \mathbf{r}')\, dV = -\mu$$

Now the integral of the term k^2R^2A, which is proportional to R^2, will vanish as r_0 tends to zero. Hence, for sufficiently small r_0,

$$\int_V \nabla^2 A\, dV = -\mu$$

Since $\nabla^2 A = \nabla \cdot \nabla\mathbf{A}$, the divergence theorem may be used to give

$$\int_V \nabla^2 A\, dV = \oint_S \nabla A \cdot d\mathbf{S} = \oint_S \nabla A \cdot \mathbf{a}_r r_0^2 \, d\Omega$$

since $d\mathbf{S} = \mathbf{a}_r r_0^2\, d\Omega$, where $d\Omega$ is an element of solid angle. Since A is a function of R only, $\nabla A = \mathbf{a}_r(\partial A/\partial R)$, and hence

$$r_0^2 \oint_S \nabla A \cdot \mathbf{a}_r \, d\Omega = r_0^2 \oint_S \frac{\partial A}{\partial R}\, d\Omega = 4\pi r_0^2 \frac{\partial A}{\partial R} = -\mu$$

Evaluating $\partial A/\partial R$ for $R = r_0$ shows that

$$4\pi r_0^2 \frac{\partial A}{\partial R} = -4\pi C r_0^2 \left(\frac{jk}{r_0}e^{-jkr_0} + \frac{e^{-jkr_0}}{r_0^2}\right) = -4\pi C$$

in the limit as r_0 tends to zero. Hence $4\pi C = \mu$, or $C = \mu/4\pi$, in order for the singularity in the solution for A to correspond to that for a unit source.

The above solution for the vector potential from a unit source, namely,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}\mathbf{a} \tag{2.131}$$

is clearly a function of both the source point and field point. Since (2.131) is the solution for a unit source, it is often called a Green's function and denoted by the symbol $\mathbf{G}$ as

$$\mathbf{G}(\mathbf{r}|\mathbf{r}') = G(\mathbf{r}|\mathbf{r}')\mathbf{a} = G(x, y, z|x', y', z')\mathbf{a} = \frac{\mu}{4\pi}\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{a} \quad (2.132)$$

because, by definition, a Green's function is the solution of a differential equation for a unit source.

The vector potential from a general current distribution may now be expressed in the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_V \mathbf{J}(\mathbf{r}')\frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,dV' = \int_V \mathbf{J}(\mathbf{r}')G(\mathbf{r}|\mathbf{r}')\,dV' \quad (2.133)$$

since any current distribution $\mathbf{J}$ may be considered as a sum of weighted unit sources.

2.12 LORENTZ RECIPROCITY THEOREM

The Lorentz reciprocity theorem is one of the most useful theorems in the solution of electromagnetic problems, since it may be used to deduce a number of fundamental properties of practical devices. It provides the basis for demonstrating the reciprocal properties of microwave circuits and for showing that the receiving and transmitting characteristics of antennas are the same. It also may be used to establish the orthogonality properties of the modes that may exist in waveguides and cavities.† Another important use is in deriving suitable field expansions (analogous to a Fourier series expansion) for the fields radiated or coupled into waveguides and cavities by probes, loops, or coupling apertures.

To derive the theorem, consider a volume V bounded by a closed surface S as in Fig. 2.18. Let a current source $\mathbf{J}_1$ in V produce a field $\mathbf{E}_1, \mathbf{H}_1$, while a second source $\mathbf{J}_2$ produces a field $\mathbf{E}_2, \mathbf{H}_2$. These fields satisfy Maxwell's equations; so

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 \qquad \nabla \times \mathbf{H}_1 = j\omega\epsilon\mathbf{E}_1 + \mathbf{J}_1$$
$$\nabla \times \mathbf{E}_2 = -j\omega\mu\mathbf{H}_2 \qquad \nabla \times \mathbf{H}_2 = j\omega\epsilon\mathbf{E}_2 + \mathbf{J}_2$$

Expanding the relation $\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1)$ and using Maxwell's equation show that

$$\begin{aligned}\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) &= (\nabla \times \mathbf{E}_1) \cdot \mathbf{H}_2 - (\nabla \times \mathbf{H}_2) \cdot \mathbf{E}_1 - (\nabla \times \mathbf{E}_2) \cdot \mathbf{H}_1 + (\nabla \times \mathbf{H}_1) \cdot \mathbf{E}_2 \\ &= -\mathbf{J}_2 \cdot \mathbf{E}_1 + \mathbf{J}_1 \cdot \mathbf{E}_2 \qquad (2.134)\end{aligned}$$

†In any waveguide or cavity an infinite number of field solutions are possible. Any one solution is called a mode for the same reason that the various solutions for vibrating strings and membranes are called modes. Orthogonality of modes is discussed in Sec. 3.14.

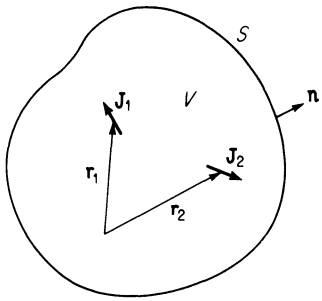

FIGURE 2.18
Illustration of the Lorentz reciprocity theorem.

Integrating both sides over the volume V and using the divergence theorem give

$$\int_V \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1)\, dV = \oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{n}\, dS$$

$$= \int_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2)\, dV \qquad (2.135)$$

where $\mathbf{n}$ is the unit outward normal to S.

Equation (2.135) is the basic form of the Lorentz reciprocity theorem.† For a number of typical situations that occur, the surface integral vanishes. If S is a perfectly conducting surface, then $\mathbf{n} \times \mathbf{E}_1 = \mathbf{n} \times \mathbf{E}_2 = 0$ on S. Since $\mathbf{E}_1 \times \mathbf{H}_2 \cdot \mathbf{n} = (\mathbf{n} \times \mathbf{E}_1) \cdot \mathbf{H}_2$, etc., the surface integral vanishes in this case. If the surface S is characterized by a surface impedance Z_m, then, according to (2.71),

$$\mathbf{E}_t = -Z_m \mathbf{n} \times \mathbf{H} \qquad \text{or} \qquad \mathbf{n} \times \mathbf{E} = -Z_m \mathbf{n} \times (\mathbf{n} \times \mathbf{H})$$

[note that in (2.71) $\mathbf{n}$ points into the region occupied by the field, and hence the minus sign is used here, since $\mathbf{n}$ is directed out of V]. Consequently,

$$\begin{aligned}(\mathbf{n} \times \mathbf{E}_1) \cdot \mathbf{H}_2 &- (\mathbf{n} \times \mathbf{E}_2) \cdot \mathbf{H}_1 \\ &= -Z_m[\mathbf{n} \times (\mathbf{n} \times \mathbf{H}_1)] \cdot \mathbf{H}_2 + Z_m[\mathbf{n} \times (\mathbf{n} \times \mathbf{H}_2)] \cdot \mathbf{H}_1 \\ &= Z_m(\mathbf{n} \times \mathbf{H}_2) \cdot (\mathbf{n} \times \mathbf{H}_1) - Z_m(\mathbf{n} \times \mathbf{H}_1) \cdot (\mathbf{n} \times \mathbf{H}_2) = 0\end{aligned}$$

and the surface integral vanishes again.

†In anisotropic media with nonsymmetric permittivity or permeability tensors, a modified form must be used. See, for example, R. F. Harrington and A. T. Villeneuve, Reciprocity Relations for Gyrotropic Media, *IRE Trans.*, vol. MTT-6, pp. 308–310, July, 1958.

Another example where the surface integral vanishes is when S is chosen as a spherical surface at infinity for which $\mathbf{n} = \mathbf{a}_r$. The radiated field at infinity is a spherical TEM wave for which

$$\mathbf{H} = Y\mathbf{a}_r \times \mathbf{E} = \left(\frac{\epsilon}{\mu}\right)^{1/2} \mathbf{a}_r \times \mathbf{E}$$

Therefore

$$(\mathbf{n} \times \mathbf{E}_1) \cdot \mathbf{H}_2 - (\mathbf{n} \times \mathbf{E}_2) \cdot \mathbf{H}_1 = Y(\mathbf{a}_r \times \mathbf{E}_1) \cdot (\mathbf{a}_r \times \mathbf{E}_2) - Y(\mathbf{a}_r \times \mathbf{E}_2) \cdot (\mathbf{a}_r \times \mathbf{E}_1) = 0$$

and the surface integral vanishes.

Actually, for any surface S which encloses all the sources for the field, the surface integral will vanish. This result may be seen by applying (2.135) to the volume V bounded by S and the surface of a sphere of infinite radius. There are no sources in this volume, and since the surface integral over the surface of the sphere with infinite radius is zero, we must have, from (2.135),

$$\oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot (-\mathbf{n})\, dS = 0 = \oint_S (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \mathbf{n}\, dS$$

Hence the surface integral taken over any closed surface S surrounding all the sources vanishes.

When the surface integral vanishes, (2.135) reduces to

$$\int_V \mathbf{E}_1 \cdot \mathbf{J}_2\, dV = \int_V \mathbf{E}_2 \cdot \mathbf{J}_1\, dV \tag{2.136}$$

If $\mathbf{J}_1$ and $\mathbf{J}_2$ are infinitesimal current elements, then

$$\mathbf{E}_1(\mathbf{r}_2) \cdot \mathbf{J}_2(\mathbf{r}_2) = \mathbf{E}_2(\mathbf{r}_1) \cdot \mathbf{J}_1(\mathbf{r}_1) \tag{2.137}$$

which states that the field $\mathbf{E}_1$ produced by $\mathbf{J}_1$ has a component along $\mathbf{J}_2$ that is equal to the component along $\mathbf{J}_1$ of the field radiated by $\mathbf{J}_2$ when $\mathbf{J}_1$ and $\mathbf{J}_2$ have unit magnitude. The form (2.137) is essentially the reciprocity principle used in circuit analysis except that $\mathbf{E}$ and $\mathbf{J}$ are replaced by the voltage V and current I. The applications of the reciprocity theorem are illustrated at various points throughout the text and hence are not discussed further at this time.

PROBLEMS

2.1. An atom of atomic number Z has a nuclear charge Ze and Z electrons revolving around it. As a model of this atom, consider the nucleus as a point charge and treat the electron cloud as a total charge $-Ze$ distributed uniformly throughout a sphere of radius r_0. When an external field E is applied, the nucleus is displaced an amount x. Show that a restoring force $x(Ze)^2/4\pi r_0^3\epsilon_0$ is produced and must be equal to ZeE. Thus show that the induced dipole moment is $p = 4\pi\epsilon_0 r_0^3 E$ and is linearly related to E.

2.2. In a certain material the equation of motion for the polarization is

$$\frac{d^2\mathscr{P}}{dt^2} + \nu\frac{d\mathscr{P}}{dt} + \omega_0^2\mathscr{P} = 2\epsilon_0\omega_0^2\mathscr{E}$$

where $\mathscr{E}$ is the total field in the dielectric. Find the relation between $\mathscr{P}$ and $\mathscr{E}$ when $\mathscr{E} = \text{Re}(Ee^{j\omega t})$ and E is real. If $\omega_0 = 10^{11}$ and $\nu = 10^{10}$, over what frequency range can a relationship such as $\mathscr{D} = \epsilon\mathscr{E} = \epsilon_0\mathscr{E} + \mathscr{P}$ be written if it is assumed that the criterion to be used is that the phase difference between $\mathscr{D}$ and $\mathscr{E}$ should not exceed 5°? Plot the magnitude and phase angle of the dielectric constant $\epsilon_r = \epsilon/\epsilon_0 = (\epsilon' - j\epsilon'')/\epsilon_0$ as a function of ω.

2.3. A dielectric material is characterized by a matrix (tensor) permittivity

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \frac{\epsilon_0}{4}\begin{bmatrix} 7 & 3 & -2\sqrt{0.5} \\ 3 & 7 & -2\sqrt{0.5} \\ -2\sqrt{0.5} & -2\sqrt{0.5} & 10 \end{bmatrix}$$

when referred to the xyz coordinate frame. If the coordinate axis is rotated into the principal axis u, v, w, the permittivity is exhibited in diagonal form:

$$[\epsilon] = \begin{bmatrix} \epsilon_{uu} & 0 & 0 \\ 0 & \epsilon_{vv} & 0 \\ 0 & 0 & \epsilon_{ww} \end{bmatrix}$$

Find the principal axis and the values of the principal dielectric constants ϵ_{uu}/ϵ_0, etc.

Hint: By definition, along a principal axis a scalar equation such as $D_u = \epsilon_{uu}E_u$ holds. In general, if $\mathbf{D}$ is directed along a principal axis, then

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \frac{\epsilon_0}{4}\begin{bmatrix} 7 & 3 & -2\sqrt{0.5} \\ 3 & 7 & -2\sqrt{0.5} \\ -2\sqrt{0.5} & -2\sqrt{0.5} & 10 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \lambda\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

or in words, when $\mathbf{D}$ is directed along a principal axis, it is related to $\mathbf{E}$ by a scalar constant λ. The above constitutes a set of three homogeneous equations, of which the first is

$$\left(\frac{7\epsilon_0}{4} - \lambda\right)E_x + \frac{3\epsilon_0}{4}E_y - \frac{2\epsilon_0\sqrt{0.5}}{4}E_z = 0$$

Verify that, for a solution, the following determinant must vanish:

$$\begin{vmatrix} 7-4\lambda/\epsilon_0 & 3 & -2\sqrt{0.5} \\ 3 & 7-4\lambda/\epsilon_0 & -2\sqrt{0.5} \\ -2\sqrt{0.5} & -2\sqrt{0.5} & 10-4\lambda/\epsilon_0 \end{vmatrix} = 0$$

This cubic equation gives three roots for λ, which may be identified as $\epsilon_{uu}, \epsilon_{vv}, \epsilon_{ww}$. For any one root, say ϵ_{uu}, the components of a vector directed along the corresponding principal axis are proportional to the cofactors of the above determinant. This type of problem is called a matrix eigenvalue problem. The λ's are the eigenvalues.

Answer: $\epsilon_{uu} = 3\epsilon_0$, $\epsilon_{vv} = 2\epsilon_0$, $\epsilon_{ww} = \epsilon_0$.

Unit vectors along the principal axis are

$$\mathbf{a}_u = 0.5\mathbf{a}_x + 0.5\mathbf{a}_y - \sqrt{0.5}\,\mathbf{a}_z$$

$$\mathbf{a}_v = 0.5\mathbf{a}_x + 0.5\mathbf{a}_y + \sqrt{0.5}\,\mathbf{a}_z$$

$$\mathbf{a}_w = \sqrt{0.5}\,\mathbf{a}_x - \sqrt{0.5}\,\mathbf{a}_y$$

2.4. In the interior of a medium with conductivity σ and permittivity ϵ, free charge is distributed with a density $\rho_0(x,y,z)$ at time $t = 0$. Show that the charge decays according to

$$\rho = \rho_0 e^{-t/\tau} \qquad \tau = \frac{\epsilon}{\sigma}$$

Evaluate the relaxation time τ for copper for which $\sigma = 5.8 \times 10^7$ S/m, $\epsilon = \epsilon_0$. Find τ for sea water also for which $\sigma = 4$ S/m and $\epsilon = 80\epsilon_0$. If the relaxation time is short compared with the period of an applied time-harmonic field, there will be negligible accumulation of free charge and $\nabla \cdot \mathbf{D}$ may be assumed to be zero. What is the upper frequency limit for which this is true in the case of copper and sea water, i.e., the frequency for which τ is equal to the period?

Hint: Use the continuity equation, Ohm's law, and the divergence equation for **D**.

2.5. Show that, when the relaxation time for a material is small compared with the period of the time-harmonic field, the displacement current may be neglected in comparison with the conduction current.

2.6. Consider two concentric spheres of radii a and b. The outer sphere is kept at a potential V, and the inner sphere at zero potential. Solve Laplace's equation in spherical coordinates to find the potential and electric field between the spheres. Take $b > a$.

2.7. Solve Laplace's equation to find the potential and electric field between two coaxial cylinders of radii a and b if the center cylinder is kept at a potential V and the outer cylinder at zero potential. Take $b > a$.

2.8. Derive (2.45) from (2.18).

2.9. Derive (2.47).

2.10. Express the scalar Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ in cylindrical coordinates. If $\psi = f(\phi)g(r)h(z)$, find the differential equations satisfied by f, g, and h.

2.11. When material polarization $\mathscr{P}$ and $\mathscr{M}$ are explicitly taken into account, show that the wave equations satisfied by $\mathscr{E}$ and $\mathscr{H}$ are

$$\nabla^2\mathscr{H} - \mu_0\epsilon_0\frac{\partial^2\mathscr{H}}{\partial t^2} = -\nabla\nabla\cdot\mathscr{M} + \mu_0\epsilon_0\frac{\partial^2\mathscr{M}}{\partial t^2} - \frac{\partial}{\partial t}\nabla\times\mathscr{P} - \nabla\times\mathscr{J}$$

$$\nabla^2\mathscr{E} - \mu_0\epsilon_0\frac{\partial^2\mathscr{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathscr{P}}{\partial t^2} + \mu_0\frac{\partial\mathscr{J}}{\partial t} + \mu_0\frac{\partial}{\partial t}\nabla\times\mathscr{M} + \frac{1}{\epsilon_0}\nabla\rho - \frac{\nabla\nabla\cdot\mathscr{P}}{\epsilon_0}$$

Note that $\nabla\cdot\mathscr{B} = 0$; so $\nabla\cdot\mathscr{H} = -\nabla\cdot\mathscr{M}$ and $\nabla\cdot\mathscr{D} = \rho$; so $\nabla\cdot\epsilon_0\mathscr{E} = \rho - \nabla\cdot\mathscr{P}$. Examination of the source terms in the above equations shows that $\partial\mathscr{P}/\partial t$ is a polarization current analogous to conduction current $\mathscr{J}$.

2.12. Derive (2.62).

2.13. Between two perfectly conducting coaxial cylinders of radii a and b, $b > a$, the electromagnetic field is given by

$$\mathbf{E} = \mathbf{a}_r E_0 r^{-1} e^{-jk_0 z} \qquad H = \mathbf{a}_\phi Y_0 E_0 r^{-1} e^{-jk_0 z}$$

where $k_0 = \omega(\mu_0\epsilon_0)^{1/2}$, $Y_0 = (\epsilon_0/\mu_0)^{1/2}$. Find the potential difference between the cylinders and the total current on the inner and outer cylinders. Express the power in terms of the voltage and current, and show that it is equal to that computed from an integration of the complex Poynting vector over the coaxial-line cross section. Show that the characteristic impedance of the line is $V/I = (Z_0/2\pi)\ln(b/a) = 60\ln(b/a)$, where V is the voltage and I is the total current on one cylinder.

2.14. A round wire of radius r_0 much greater than the skin depth δ_s has a uniform electric field E applied in the axial direction at its surface. Use the surface-impedance concept to find the total current on the wire. Show that the ratio of the ac impedance of the wire to the dc resistance is

$$\frac{Z_{ac}}{R_{dc}} = \frac{r_0\sigma}{2}Z_m$$

Evaluate this ratio for copper at $f = 10^6$ Hz for $\sigma = 5.8 \times 10^7$ S/m, $r_0 = 0.1$ cm, $\mu = \mu_0$.

2.15. The half-space $z \ge 0$ is filled with a material with permittivity ϵ_0 and permeability $\mu \ne \mu_0$. A parallel polarized plane TEM wave is incident at an angle θ_1, as in Fig. 2.14. Find the reflection and transmission coefficients for the electric field. Does a Brewster angle exist for which the reflection coefficient vanishes?

2.16. Repeat Prob. 2.15 for the case of a perpendicular polarized incident wave. Does a Brewster angle exist? If so, obtain an expression for it.

2.17. The half-space $z \ge 0$ is filled with a material with permeability μ and permittivity ϵ. When a plane wave is incident normally on this material, show that the reflection and transmission coefficients are

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} \qquad T = 1 + \Gamma = \frac{2Z}{Z + Z_0}$$

where $Z = (\mu/\epsilon)^{1/2}$, $Z_0 = (\mu_0/\epsilon_0)^{1/2}$. Choose an electric field with an x component only.

2.18. The half-space $z \geq 0$ is filled with a material of permittivity ϵ_2 and with $\mu = \mu_0$. A second sheet with permittivity ϵ_1 is placed in front. A plane wave is incident normally on the structure from the left, as illustrated. Verify that the reflection coefficient at the first interface vanishes if $\epsilon_1 = (\epsilon_2\epsilon_0)^{1/2}$ and the thickness $d = \frac{1}{4}\lambda_0(\epsilon_0/\epsilon_1)^{1/2}$. The electric field may be assumed to have an x component only. The matching layer is known as a quarter-wave transformer (actually an impedance transformer). This matching technique is used to reduce reflections from optical lenses and is called *lens blooming*, or *coated lenses*.

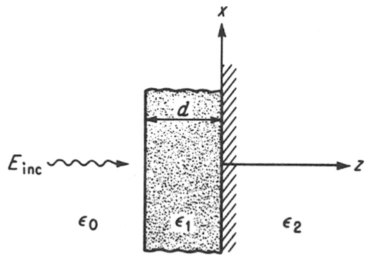

FIGURE P2.18

2.19. In terms of the vector potential $\mathbf{A}$ from a short current element $\Delta z I_0 \mathbf{a}_z$ located at the origin, show that the radiated electric and magnetic fields are

$$\mathbf{H} = \frac{I_0\,\Delta z}{4\pi}\left(\frac{jk_0}{r} + \frac{1}{r^2}\right)\mathbf{a}_\phi \sin\theta\, e^{-jk_0 r}$$

$$\mathbf{E} = -\frac{I_0\,\Delta z}{2\pi}\frac{jZ_0}{k_0}\left(\frac{jk_0}{r^2} + \frac{1}{r^3}\right)\mathbf{a}_r \cos\theta\, e^{-jk_0 r}$$

$$-\frac{I_0\,\Delta z}{4\pi}\frac{jZ_0}{k_0}\left(\frac{-k_0^2}{r} + \frac{jk_0}{r^2} + \frac{1}{r^3}\right)\mathbf{a}_\theta \sin\theta\, e^{-jk_0 r}$$

Hint: Use (2.122) and (2.121), and express $\mathbf{A}$ as components in a spherical coordinate system r, θ, ϕ. Note that $\mathbf{a}_z = \mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta$.

2.20. A dielectric may be characterized by its dipole polarization $\mathbf{P}$ per unit volume. If $\rho = \mathbf{J} = 0$ and $\mathbf{P}$ is taken into account explicitly, show that, if a vector potential $\mathbf{A}$ is introduced according to $\mathbf{B} = \nabla \times \mathbf{A}$, then $\mathbf{A}$ is a solution of

$$\nabla^2\mathbf{A} + k_0^2\mathbf{A} = -j\omega\mu_0\mathbf{P}$$

and that the fields are given by

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = \frac{\nabla\nabla\cdot\mathbf{A} + k_0^2\mathbf{A}}{j\omega\mu_0\epsilon_0}$$

Note that a Lorentz condition is used. Thus an electric dipole $\mathbf{P}$ is equivalent to a current element $j\omega\mathbf{P}$, or alternatively, a current element $\mathbf{J}$ may be considered as an electric dipole $\mathbf{P} = \mathbf{J}/j\omega$.

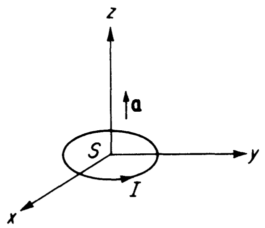

FIGURE P2.21

2.21. A small current loop constitutes a magnetic dipole $\mathbf{M} = IS\mathbf{a}$, where I is the current, S the area of the loop, and $\mathbf{a}$ a vector normal to the plane of the loop and pointing in the direction that a right-hand screw, rotating in the direction of the current, would advance. The field radiated by such a current loop, with linear dimensions much smaller than a wavelength, may be obtained by a potential theory analogous to that given in Prob. 2.20 by treating the loop as a magnetic dipole $\mathbf{M}$. Thus replace $\mathbf{B}$ by $\mu_0\mathbf{H} + \mu_0\mathbf{M}$ in Maxwell's equation and treat $\mathbf{M}$ as a source term. Since ρ is zero, $\nabla \cdot \mathbf{D} = 0$, and this permits $\mathbf{D}$ to be expressed as $\mathbf{D} = -\nabla \times \mathbf{A}_m$, where $\mathbf{A}_m$ is a magnetic-type vector potential. By paralleling the development in the text for the potential $\mathbf{A}$, show that the following relations are obtained:

$$\nabla^2\mathbf{A}_m + k_0^2\mathbf{A}_m = -j\omega\mu_0\epsilon_0\mathbf{M}$$

$$\mathbf{D} = -\nabla \times \mathbf{A}_m$$

$$\mathbf{H} = \frac{k_0^2 A_m + \nabla\nabla \cdot \mathbf{A}_m}{j\omega\mu_0\epsilon_0}$$

Hence, for a z-directed magnetic dipole at the origin,

$$\mathbf{A}_m = \frac{j\omega\mu_0\epsilon_0\mathbf{M}}{4\pi r}e^{-jk_0 r}$$

from which the fields are readily found. Note that in this problem $\mathbf{M}$ represents the magnetic dipole source density in Maxwell's equations, but in the solution for the vector potential it represents the total magnetic dipole strength. It would have been more consistent to use $\mathbf{M}\,\delta(\mathbf{r} - \mathbf{r}')$ to represent the source density, where $\delta(\mathbf{r} - \mathbf{r}')$ is the three-dimensional Dirac delta function which has the property

$$\int_V \delta(\mathbf{r} - \mathbf{r}')\, dV' = 1 \qquad \mathbf{r} \text{ in } V$$

2.22. Consider an arbitrary current element $\mathbf{J}_1$ in front of a perfectly conducting plane. This current radiates a field $\mathbf{E}_1$ having zero tangential components on the conducting plane. Use the Lorentz reciprocity theorem to show that a current $\mathbf{J}_2$ parallel to the conducting plane and an infinitesimal distance in front of it does not radiate.

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