APPENDIX I

USEFUL RELATIONS FROM VECTOR ANALYSIS

I.1 VECTOR ALGEBRA

Let vectors **A** and **B** be expressed as components along unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ in a right-hand orthogonal coordinate system. Then

$$\mathbf{A} \pm \mathbf{B} = (A_1 \pm B_1)\mathbf{a}_1 + (A_2 \pm B_2)\mathbf{a}_2 + (A_3 \pm B_3)\mathbf{a}_3 \tag{I.1}$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta = A_1B_1 + A_2B_2 + A_3B_3 \tag{I.2}$$

where θ is the angle between **A** and **B**.

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_1(A_2B_3 - A_3B_2) + \mathbf{a}_2(A_3B_1 - A_1B_3) + \mathbf{a}_3(A_1B_2 - A_2B_1) \tag{I.3a}$$

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\sin\theta \tag{I.3b}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \times \mathbf{A} \cdot \mathbf{B} \tag{I.4}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{I.5}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{I.6}$$

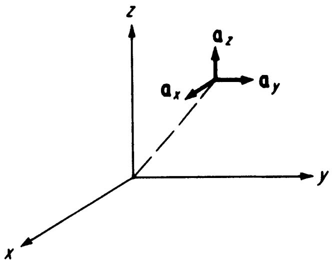

FIGURE I.1
Rectangular coordinates.

I.2 VECTOR OPERATIONS IN COMMON COORDINATE SYSTEMS

Rectangular Coordinates

$$\nabla \Phi = \mathbf{a}_x \frac{\partial \Phi}{\partial x} + \mathbf{a}_y \frac{\partial \Phi}{\partial y} + \mathbf{a}_z \frac{\partial \Phi}{\partial z} \tag{I.7}$$

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{I.8}$$

$$\operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{I.9}$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{I.10}$$

$$\nabla^2 \mathbf{A} = \mathbf{a}_z \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z \tag{I.11}$$

Cylindrical Coordinates

$$\nabla \Phi = \mathbf{a}_r \frac{\partial \Phi}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial \Phi}{\partial \phi} + \mathbf{a}_z \frac{\partial \Phi}{\partial z} \tag{I.12}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{I.13}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \mathbf{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{a}_z \left[\frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right] \tag{I.14}$$

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{I1.5}$$

$$\nabla^2 \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A} \tag{I.16}$$

Note that $\nabla^2 \mathbf{A} \neq \mathbf{a}_r \nabla^2 A_r + \mathbf{a}_\phi \nabla^2 A_\phi + \mathbf{a}_z \nabla^2 A_z$ since $\nabla^2 \mathbf{a}_r A_r \neq \mathbf{a}_r \nabla^2 A_r$, etc., because the orientation of the unit vectors $\mathbf{a}_r, \mathbf{a}_\phi$ varies with the coordinates r, ϕ.

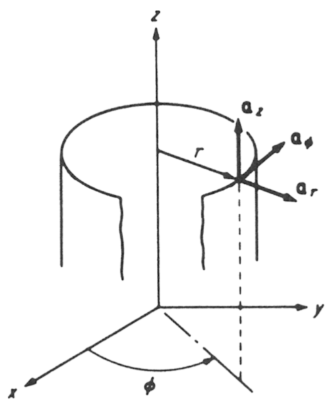

FIGURE I.2
Cylindrical coordinates.

Spherical Coordinates

$$\nabla\Phi = \mathbf{a}_r \frac{\partial \Phi}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\mathbf{a}_\phi}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} \tag{I.17}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{I.18}$$

$$\nabla\times\mathbf{A} = \frac{\mathbf{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\mathbf{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+ \frac{\mathbf{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \tag{I.19}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \tag{I.20}$$

$$\nabla^2\mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla\times\nabla\times\mathbf{A} \tag{I.21}$$

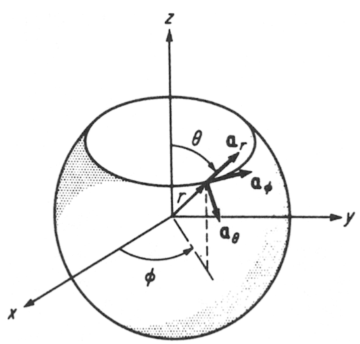

FIGURE I.3
Spherical coordinates.

I.3 VECTOR IDENTITIES

$$\nabla(\Phi\psi) = \psi\,\nabla\Phi + \Phi\,\nabla\psi \tag{I.22}$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A} \tag{I.23}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A} \tag{I.24}$$

$$\nabla \times (\psi\mathbf{A}) = (\nabla\psi) \times \mathbf{A} + \psi\nabla \times \mathbf{A} \tag{I.25}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{I.26}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{I.27}$$

$$\nabla \cdot \nabla\Phi = \nabla^2\Phi \tag{I.28}$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{I.29}$$

$$\nabla \times \nabla\Phi = 0 \tag{I.30}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla^2\mathbf{A} \tag{I.31}$$

If $\mathbf{A}$ and Φ are continuous functions with at least piecewise continuous first derivatives in V and on S (or on S and the contour C bounding S),

$$\int_V \nabla\Phi\, dV = \oint_S \Phi\, d\mathbf{S} \tag{I.32}$$

$$\int_V \nabla \cdot \mathbf{A}\, dV = \oint_S \mathbf{A} \cdot dS \qquad \text{(divergence theorem)} \tag{I.33}$$

$$\int_V \nabla \times \mathbf{A}\, dV = \oint_S \mathbf{n} \times \mathbf{A}\, dS \qquad d\mathbf{S} = \mathbf{n}\, dS \tag{I.34}$$

$$\int_S \mathbf{n} \times \nabla\Phi\, dS = \oint_C \Phi\, d\mathbf{l} \tag{I.35}$$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes' theorem)} \tag{I.36}$$

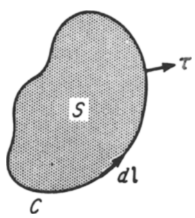

FIGURE I.4
Surface S bounded by contour C.

I.4 GREEN'S IDENTITIES

If **A**, **B**, Φ, and ψ are continuous with piecewise continuous first derivatives,

$$\int_V (\nabla\Phi \cdot \nabla\psi + \psi\,\nabla^2\Phi)\,dV = \oint_S \psi\,\nabla\Phi \cdot d\mathbf{S} \tag{I.37}$$

which is Green's first identity. Green's second identity is

$$\int_V (\psi\,\nabla^2\Phi - \Phi\,\nabla^2\psi)\,dV = \oint(\psi\,\nabla\Phi - \Phi\,\nabla\psi) \cdot d\mathbf{S} \tag{I.38}$$

In two dimensions (I.37) becomes

$$\int_S (\nabla_t\Phi \cdot \nabla_t\psi + \psi\,\nabla_t^2\Phi)\,dS = \oint_C \psi\,\nabla_t\Phi \cdot \boldsymbol{\tau}\,dl \tag{I.39}$$

where ∇_t is the del operator in two dimensions and $\boldsymbol{\tau}$ is a unit vector normal to C and in the plane of S. The two-dimensional form of (I.38) is similar.

The vector forms of Green's identities are

$$\int_V \nabla \cdot (\mathbf{A} \times \nabla \times \mathbf{B})\,dV = \int_V [(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B}]\,dV$$

$$= \oint_S \mathbf{A} \times (\nabla \times \mathbf{B}) \cdot d\mathbf{S} \tag{I.40}$$

$$\int_V (\mathbf{B} \cdot \nabla \times \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{B})\,dV$$

$$= \oint_S [\mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A})] \cdot d\mathbf{S} \tag{I.41}$$